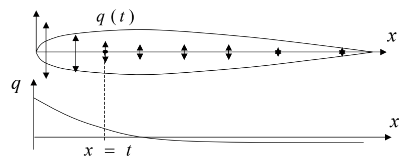
Line Source Distributions

- In the previous section we say a numerical approach to representing a surface by a large number of sources.
- In the limit as the number approaches infinite, rather than discrete sources, we get a continuous distribution.
- To represent this Line Source distribution, we introduce the source strength per unit length by:

$$q(t)dt \equiv \text{net strength between x = t and x = t + dt}$$



Line Source Distributions [2]

- Rather than the summation we had for discrete sources, the total effect of a line source is found by integration.
- Thus, for a line source running from leading to trailing edge of an airfoil, the stream and potential functions are:

$$\psi_s(x,y) = \int_0^c \frac{q(t)}{2\pi}\theta dt = \int_0^c \frac{q(t)}{2\pi}\text{atan}\left(\frac{y}{x-t}\right)dt$$

$$\phi_s(x,y) = \int_0^c \frac{q(t)}{2\pi}\ln(r)dt = \int_0^c \frac{q(t)}{2\pi}\ln\left(\sqrt{(x-t)^2+y^2}\right)dt$$

- These can still be combined with a uniform horizontal flow: $\psi = V_\infty y + \psi_s(x,y)$ $\quad \phi = V_\infty x + \phi_s(x,y)$

Thin Airfoil Theory – Symmetric

- In Section 3.5 of the book, Moran discusses how to use these line distributions to model surfaces.
- This is potentially a very complex task involving complex integrations.
- However, some closed form mathematical solutions are possible if the Thin Airfoil assumptions are used.
- To see this, first express the local velocities as the combination of the freestream plus local perturbations due to the line source:

$$u(x,y) = V_\infty + u_s(x,y) \qquad v(x,y) = v_s(x,y)$$

- Then our surface boundary condition is:

$$\frac{v_s(x,Y)}{V_\infty + u_s(x,Y)} = \frac{dY(x)}{dx}$$

Thin Airfoil Theory – Symmetric [2]

- However, if a body is very thin, then at most locations:

$$V_\infty >> u_s \quad \text{or} \quad v_s$$

- Thus, we can approximate the boundary condition by:

$$\boxed{\frac{v_s(x,Y)}{V_\infty} \approx \frac{dY(x)}{dx}}$$

- Similarly, the pressure coefficient given by:

$$c_P = 1 - \left(\frac{V_\infty + u_s}{V_\infty}\right)^2 + \left(\frac{v_s}{V_\infty}\right)^2 = -2\frac{u_s}{V_\infty} - \left(\frac{u_s}{V_\infty}\right)^2 + \left(\frac{v_s}{V_\infty}\right)^2$$

- Can be approximated by:

$$\boxed{c_P \approx -2\frac{u_s}{V_\infty}}$$

Thin Airfoil Theory – Symmetric [3]

- These assumptions are obviously going to breakdown near the leading edge stagnation point where:

$$V = V_\infty + u_s = 0 \qquad \Rightarrow \qquad u_s = -V_\infty$$

- Not as obvious is the fact that they also breakdown at the trailing edge – where there is another stagnation point.
- This aft stagnation point is obvious on cylinder, oval or ellipse shapes – but is much less pronounced on sharp trailing edge airfoils.
- In real (i.e. viscous) flow, the boundary layer swallows this point within it, so it never really exists at all.

Thin Airfoil Theory – Symmetric [4]

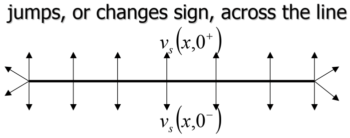
- Let's for a moment ignore the problem at the stagnation points which we will revisit.
- Instead, lets make one more approximation which takes a leap of faith.
- Since we are considering only thin bodies, rather than apply the boundary condition on the surface, let's apply it on the $y=0$ axis, or:

$$\boxed{v_s(x,0) \approx V_\infty \frac{dY(x)}{dx}}$$

- At first glance this might seem rediculous since we expect by symmetry that $v=0$ at $y=0$.
- But, consider what a line source looks like.

Thin Airfoil Theory – Symmetric [5]

- Since a source is spitting out mass, the vertical velocity jumps, or changes sign, across the line source.



$$v_s(x,0^+) = -v_s(x,0^-)$$

- Note also that there isn't an infinite velocity on the line since there is not a radial change in flow area!
- Since we are still only doing symmetric bodies, we only have to deal with one side right now – later, on lifting airfoils, we will have to deal with asymmetry.

Thin Airfoil Theory – Symmetric [6]

- And, if we see how we are going to evaluate the local velocity due to the line source at $y=0$, we get:

$$v_s(x,0) = \frac{1}{2\pi} Lim_{y\to 0}\left[\int_0^c \frac{q(t)y}{(x-t)^2+y^2}\,dt\right]$$

- While the numerator of the integrand will go to zero, realize that the denominator will go to zero also near where $x=t$.
- Choose a small region around $x=t$, say from $x-\varepsilon$ to $x+\varepsilon$ (ε is very small), we can assume $q(t)$ is constant in this interval and equal to $q(x)$.
- Everywhere else the integrand is zero.

Thin Airfoil Theory – Symmetric [7]

- Thus:

$$\begin{aligned} v_s(x,0) &= \frac{q(x)}{2\pi} Lim_{y\to 0}\left[\int_{x-\varepsilon}^{x+\varepsilon} \frac{y}{(x-t)^2+y^2}\,dt\right] \\ &= \frac{q(x)}{2\pi} Lim_{y\to 0}\left[\operatorname{atan}\left(\frac{\varepsilon}{y}\right) - \operatorname{atan}\left(\frac{-\varepsilon}{y}\right)\right] \\ &= \tfrac{1}{2} q(x) \end{aligned}$$

- When we combine this with the flow tangency boundary condition, we have a solution for $q(x)$:

$$\boxed{q(x) = 2v_s(x,0) = 2V_\infty \frac{dY(x)}{dx}}$$

Thin Airfoil Theory – Symmetric [8]

- To get the pressure distribution, we first need to evaluate the u_s velocity component, also at $y=0$:

$$u_s(x,0)=\frac{1}{2\pi}Lim_{y\to 0}\left[\int_0^c \frac{q(t)(x-t)}{(x-t)^2+y^2}dt\right]$$

- The integrand in this equation has a singularity (goes to infinity) at $x=t$, if y is also equal to zero.
- Dealing with this singularity require some complex math tricks.
- The general idea is that this integral, unlike the previous one for v_s, does not depend upon the value at $x=t$ since the integrand is odd about that point (changes from positive to negative).

Thin Airfoil Theory – Symmetric [9]

- Instead, the values of the integral upstream and downstream are what is important.
- The way to express this integration which excludes a singularity, we use the Cauchy Principal Value integral.

$$u_s(x,0)=\frac{1}{2\pi}Lim_{y\to 0}\left[\int_0^{c} \mathbf{P}\frac{q(t)(x-t)}{(x-t)^2+y^2}dt\right]$$

- After taking the limit and substiting the line strength as a function of surface slope, this becomes:

$$u_s(x,0)=\frac{V_\infty}{\pi}\int_0^c \mathbf{P}\frac{Y'(x)}{(x-t)}dt$$

Thin Airfoil Theory – Symmetric [10]

- While this last equation does not seem to useful in integral form, for many possible functions for Y', closed form solutions are possible using Appendix B.
- So, to summarize, by thin airfoil theory on symmetric bodies:

$$v_s(x,0)=\tfrac{1}{2}q(x)=V_\infty Y'(x)$$

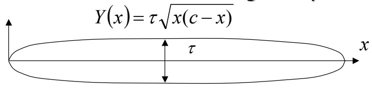
$$u_s(x,0)=\frac{V_\infty}{\pi}\int_0^c \mathbf{P}\frac{Y'(x)}{(x-t)}dt$$

$$c_P(x)=-2\frac{u_s(x,0)}{V_\infty}$$

Thin Airfoil Theory – Example 1

- To show how this analysis can work in practice, consider an elliptical airfoil whose surface is given by:

$$Y(x) = \tau\sqrt{x(c-x)}$$



- The thickness of this airfoil is τ. Also, the surface slope is given by:

$$Y'(x) = \frac{\tau}{2}\frac{c-2x}{\sqrt{x(c-x)}}$$

- Thus, the source strengths are:

$$q(x) = 2Y'(x) = \tau V_\infty \frac{c-2x}{\sqrt{x(c-x)}}$$

Thin Airfoil Theory – Example 1 [2]

- Before attempting to integrate $Y'(x)$ to find u_s, lets first introduce a change of variables which will make life easier:

$$x = \frac{c}{2}(1-\cos\theta_0)$$

- The source distribution then becomes:

$$Y'(\theta_0) = \frac{\tau}{2}\frac{c - 2\left[\frac{c}{2}(1-\cos\theta_0)\right]}{\sqrt{\left[\frac{c}{2}(1-\cos\theta_0)\right]\left\{c-\left[\frac{c}{2}(1-\cos\theta_0)\right]\right\}}}$$

$$= \frac{\tau}{2}\frac{c\cos\theta_0}{\sqrt{\left(\frac{c}{2}\right)^2(1-\cos^2\theta_0)}} = \tau\frac{\cos\theta_0}{\sin\theta_0}$$

Thin Airfoil Theory – Example 1 [3]

- For the integration, we also need to change the dummy variable of integration to:

$$t = \frac{c}{2}(1-\cos\theta) \qquad dt = \frac{c}{2}\sin\theta d\theta$$

- Finally, the limits of integration change since:

$$t = 0 \;\Rightarrow\; \theta = 0 \qquad t = c \;\Rightarrow\; \theta = \pi$$

- Thus, the integration becomes:

$$u_s(x,0) = \frac{\tau V_\infty}{\pi}\, P\!\int_0^\pi \frac{\cos\theta}{(\cos\theta - \cos\theta_o)}d\theta$$

- Which, using Appendix B, is just:

$$\boxed{u_s(x,0) = \tau V_\infty}$$

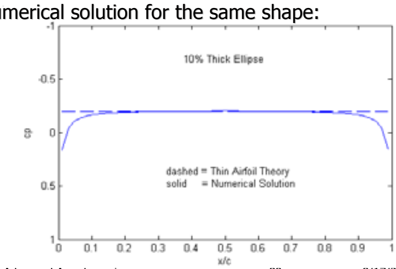
- Note that this is a constant velocity!

Thin Airfoil Theory – Example 1 [4]

- The pressure distribution is also constant:

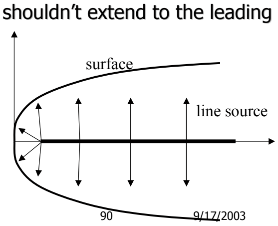
$$c_p(x) = -2\tau$$

- The plot below show a comparison of this result to a numerical solution for the same shape:



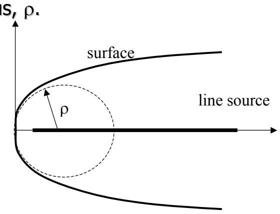
Thin Airfoil Theory – Example 1 [5]

- The plot shows that, surprisingly perhaps, thin airfoil does a good job of predicting the pressure distribution.
- The error at the leading and trailing edge, the stagnation points, is understandable given the assumptions.
- However, one way to explain the discrepancy is the fact that the source really shouldn't extend to the leading edge:
- If the line source started a little to the right of the y axis, a stagnation point would be formed.



Thin Airfoil Theory – Example 1 [6]

- Moran states the to model the leading edge, the source should be offset by an amount equal to half the leading edge radius, ρ.

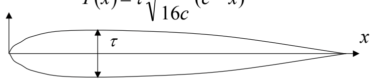


- In practice, we will just live with the error since the leading edge region is usually pretty small.

Thin Airfoil Theory – Example 2

- The second example Moran shows is for a very simple airfoil shape define by:

$$Y(x) = \tau\sqrt{\frac{27x}{16c}}(c-x)$$

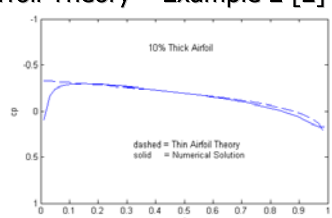


- The solution in this case is:

$$c_P(x) = -\frac{\tau}{\pi}\sqrt{\frac{27}{16}}\left[\left(\sqrt{\frac{c}{x}} - 3\sqrt{\frac{x}{c}}\right)\ln\frac{\sqrt{c}+\sqrt{x}}{\sqrt{c}-\sqrt{x}} + 6\right]$$

- This solution is compared to a numerical solution on the next slide.

Thin Airfoil Theory – Example 2 [2]



- Once again, the match is pretty good, except for leading and trailing edges.
- But, the complexity of the solution (and the math) has gone way up, and this is a very simple geometry.