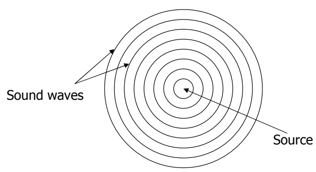
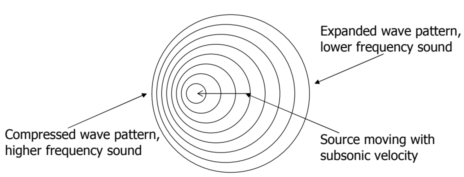
Intro to 2D Supersonic Flow

- Consider a sound source at rest in air
  - Sound waves are emitted by the source and propagate uniformly in all directions.
  - The frequency of the sound we would here depends upon the density of the sound waves



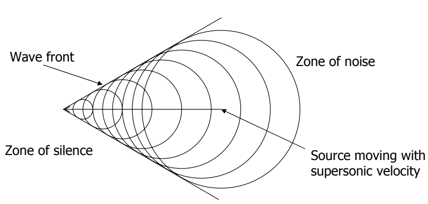
Intro to 2D Supersonic Flow [2]

- Now put the sound source into a slow motion to the left:
  - Because the speed of the wave is independent of the speed of the source, we see the waves bunching up in front.
  - The frequency of the sound is no longer uniform in all directions, but varies from in front to behind



Mach Waves

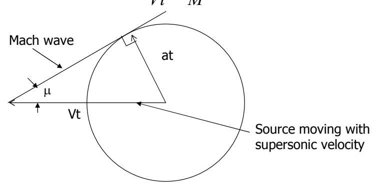
- If we increase the source speed to supersonic:
  - Source is now moving faster than the sound it emitted
  - A wave front is formed from the locus of sound waves: only behind this wave front is the source heard.
  - This wave front has another name: a **Mach wave**



Mach Waves [2]

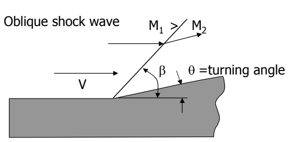
- The angle the Mach wave makes with the source velocity vector is called the Mach angle, μ.
- By using the length of the two velocity paths we see that:

$$\sin\mu = \frac{at}{Vt} = \frac{1}{M}$$



Oblique Shock Waves

- A Mach wave is the very weak form of a more interesting phenomena, the **oblique shock wave**.
- Oblique shock waves form when a flow turns into itself. They act to turn the flow parallel to the surface and slow it down.
- The angle of an oblique shock wave is given the symbol β and is always greater than μ.

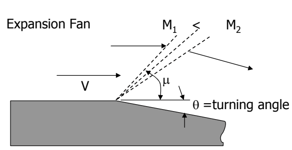


$\beta > \mu$

(But, $\beta \to \mu$ for very small turning angles)

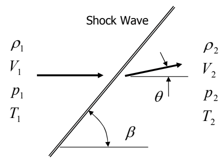
Expansion Fans

- The opposite of an oblique shock wave is called an **expansion fan**.
- Oblique shock waves form when a flow turns away from itself. They also act to turn the flow parallel to the surface but this time they speed it up.
- The expansion fan is actually made up of a series of Mach waves – thus it does not produce abrupt flow changes.



Oblique Shock Relations

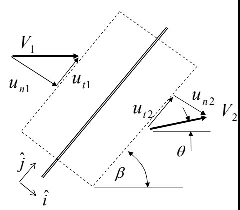
- Let's first develop relations for oblique shocks before tackling expansion fans.
- As before, assume we know all the conditions before the shock.
- In addition, we will know what the disturbance is – i.e. the value of the turning angle, θ.



- What we want to find are the post-shock conditions as well as the shock wave angle.

Oblique Shock Relations [2]

- Begin by selecting a control volume (C.V.) on which to apply our flow conservation equations.
- While any C.V. will work, one with side tangent and normal to the shock, as shown, is convenient.
- The initial and final velocities for this case have tangent and normal components:



$$u_{n1} = V_1 \sin\beta \qquad u_{t1} = V_1 \cos\beta$$

$$u_{n2} = V_2 \sin(\beta - \theta) \qquad u_{t2} = V_2 \cos(\beta - \theta)$$

Oblique Shock Relations [3]

- Now apply conservation of mass.
- Note that due to symmetry, the in- and out-flow fluxes in the tangent direction cancel each other.
- Thus, there are only normal contributions:

$$\oiint_S \rho \overline{V} \cdot \hat{n} dS = 0$$

$$-\rho_1 u_{n1} A + \rho_2 u_{n2} A = 0$$

$$\boxed{\rho_1 u_{n1} = \rho_2 u_{n2}}$$

- Next, apply momentum conservation, which in vector form is:

$$\oiint_S \rho \overline{V} (\overline{V} \cdot \hat{n}) dS + \oiint_S p \hat{n} dS = 0$$

Oblique Shock Relations [4]

- This needs to be split into tangential and normal equations.
- To get the tangential equation, dot the vector equation with the j normal vector:

$$\oiint_S \rho u_t \left(\overline{V} \cdot \hat{n}\right) dS + \oiint_S p\left(\hat{j} \cdot \hat{n}\right) dS = 0$$

$$-\rho_1 u_{n1} u_{t1} A + \rho_2 u_{n2} u_{t2} A = 0$$

$$\boxed{u_{t1} = u_{t2}}$$

- Thus, the tangential velocity doesn't change across the shock.
- This result may have been logically anticipated since there is no pressure gradient in the tangent direction.

Oblique Shock Relations [5]

- To get the normal momentum equation, dot the vector equation with the i normal vector:

$$\oiint_S \rho u_n \left(\overline{V} \cdot \hat{n}\right) dS + \oiint_S p\left(\hat{i} \cdot \hat{n}\right) dS = 0$$

$$-\rho_1 u_{n1}^2 A + \rho_2 u_{n2}^2 A - p_1 A + p_2 A = 0$$

$$\boxed{p_1 + \rho_1 u_{n1}^2 = p_2 + \rho_2 u_{n2}^2}$$

- And finally, apply energy conservation:

$$\oiint_S \rho\left(e + \tfrac{1}{2} V^2\right) \overline{V} \cdot \hat{n} dS + \oiint_S p\left(\overline{V} \cdot \hat{n}\right) dS = 0$$

- As with the earlier terms, only the terms involving the normal fluxes don't cancel.

Oblique Shock Relations [6]

- The result is thus:

$$\rho_1\left(e_1 + \tfrac{1}{2} V_1^2\right) u_{n1} + p_1 u_{n1} = \rho_2\left(e_2 + \tfrac{1}{2} V_2^2\right) u_{n2} + p_2 u_{n2}$$

- Or, after rearranging and apply our mass conservation result:

$$\rho_1 u_{n1}\left(e_1 + p_1/\rho_1 + \tfrac{1}{2} V_1^2\right) = \rho_2 u_{n2}\left(e_2 + p_2/\rho_2 + \tfrac{1}{2} V_2^2\right)$$

$$\boxed{h_1 + \tfrac{1}{2} V_1^2 = h_2 + \tfrac{1}{2} V_2^2}$$

- This equation actually applies to all adiabatic flows.
- Another form which is useful can be found by expansion:

$$h_1 + \tfrac{1}{2}\left(u_{n1}^2 + u_{t1}^2\right) = h_2 + \tfrac{1}{2}\left(u_{n2}^2 + u_{t2}^2\right)$$

Oblique Shock Relations [7]

- But since the tangential velocity doesn't change, this is just: $h_1 + \frac{1}{2}u_{n1}^2 = h_2 + \frac{1}{2}u_{n2}^2$
- So, to summarize all the results:

$$\begin{aligned} u_{t1} &= u_{t2} \\ \rho_1 u_{n1} &= \rho_2 u_{n2} \\ p_1 + \rho_1 u_{n1}^2 &= p_2 + \rho_2 u_{n2}^2 \\ h_1 + \tfrac{1}{2}u_{n1}^2 &= h_2 + \tfrac{1}{2}u_{n2}^2 \end{aligned}$$

- But note that these last 3 equation are exactly like the normal shock equations – with the normal velocity replacing the x velocity component.
- Thus, we can use our previous shock jump results by just inserting the normal Mach number.

Oblique Shock Jump Equations

- Using the normal Mach number, $M_{1n} = M_1 \sin\beta$, we can write the oblique shock jump equations as either:

$$M_{n2}^2 = \frac{1 + \frac{(\gamma-1)}{2} M_{n1}^2}{\gamma M_{n1}^2 - \frac{(\gamma-1)}{2}}$$

$$\frac{\rho_2}{\rho_1} = \frac{u_{n1}}{u_{n2}} = \frac{(\gamma+1)M_{n1}^2}{2+(\gamma-1)M_{n1}^2}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{n1}^2 - 1\right)$$

$$M_2^2 \sin^2(\beta-\theta) = \frac{1 + \frac{(\gamma-1)}{2} M_1^2 \sin^2\beta}{\gamma M_1^2 \sin^2\beta - \frac{(\gamma-1)}{2}}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1 \sin\beta}{V_2 \sin(\beta-\theta)} = \frac{(\gamma+1)M_1^2 \sin^2\beta}{2+(\gamma-1)M_1^2\sin^2\beta}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2\sin^2\beta - 1\right)$$

- We can also use the shock jump tables – as long as M_{1n} is used when looking up values.
- But, that means we still need to know β first!

Shock Wave Angle

- To find the shock wave angle, use the following geometric relations:

$$\tan\beta = \frac{u_{n1}}{u_{t1}} \qquad \tan(\beta-\theta) = \frac{u_{n2}}{u_{t2}}$$

- Applying these to the velocity jump equation:

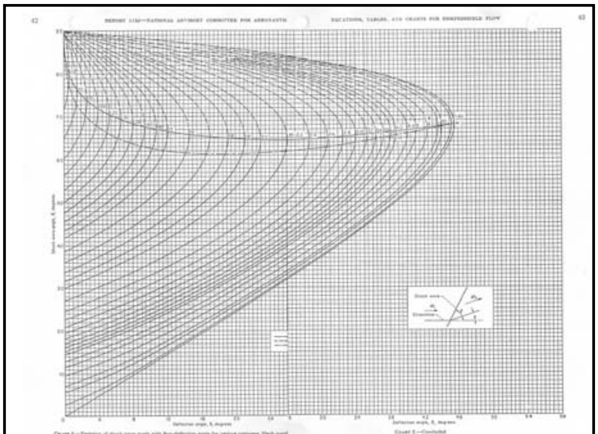
$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{u_{n2}}{u_{n1}} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}$$

- And, after using some trig identities…and a lot of work…this becomes:

$$\tan\theta = 2\cot\beta \frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos 2\beta)+2}$$

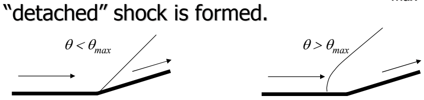
Shock Wave Angle [2]

- Unfortunately, this equation expresses θ as a function of β – the opposite of what is useful.
- Since this function is non-linear, it cannot be inverted – instead we rely upon the plot.
- Pages 513-514 show the plot θ-β-M diagram for air copied from pages 42-43 of NACA Report 1135.



Results from θ-β-M Diagram

- At any given Mach number, there is a maximum turning angle, θ_{max}, with a solution for β.
- If the body turning angle is greater than θ_{max} then a "detached" shock is formed.



- The detached shock is normal at the wall – giving subsonic flow which can make the turn.
- Detached shock are VERY difficult problems to solve – in fact I think you need CFD to do it.

Results from θ-β-M Diagram [2]

- For any given $\theta < \theta_{max}$, there actually two possible solutions!
  - The lower solution is called the "weak" solution and is the one observed in almost all external flows.
  - The upper solution is called the "strong" solution. However, this solution is unstable and doesn't normally occur except in some internal flow situations.
- For any initial Mach number M_1, the two solutions correspond to:
  - A normal shock for the strong solution.
  - A Mach wave for the weak solution, with

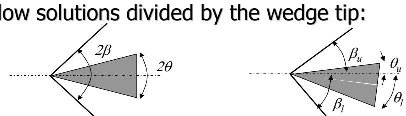
$$\beta = \mu = \sin^{-1}\frac{1}{M_1}$$

Results from θ-β-M Diagram [3]

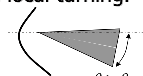
- The chart shows the line of solutions for $M_2=1.0$.
  - All the strong solution result in subsonic flow behind the shock.
  - Almost all the weak solutions have $M_2>1$, i.e. supersonic flow behind the shock!
  - There are a few weak solutions, all around θ_{max} which result in subsonic flow behind the shock.

Wedge and Cone Flow

- The previous results for a wall with ramp can also be applied to the flow over a wedge.
- The upper and lower surfaces form independent flow solutions divided by the wedge tip:

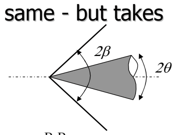


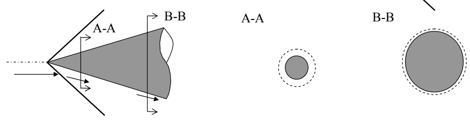
- Thus, we can apply our oblique shock relations to each surface independently using the local turning.
- Hovever, note that if $\theta > \theta_{max}$ on one side – the shock detaches on both!.



Wedge and Cone Flow [2]

- If the body is axisymmetric, i.e. a cone, rather than a wedge, the flow looks the same - but takes on a very different character.
- To see this consider sectional cuts of the flow at two locations:

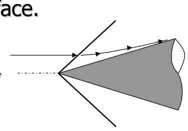




- A streamtube of air next to the surface has a growing circumference as it goes down stream.

Wedge and Cone Flow [3]

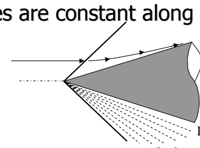
- To conserve mass, the tube thickness decreases to give close to the same flow area at all locations.
- Thus, stream lines asymptotically approach the surface of the cone – unlike the wedge where streamlines are tangent to the surface.
- Since the initial turning angle for cone flow is less than for a wedge, the shock must be weaker – and has a angle closer to μ.



- Between the shock and the cone, the flow goes through a smooth,isentropic turning and compression.

Wedge and Cone Flow [4]

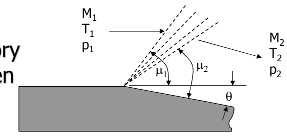
- As it turns out, rather than have constant flow properties behind the shock as in 2-D – the properties are constant along rays from the apex:



- This observation is one of the basis for conical flow theory – and approximate method for solving 3-D flows over wings and fuselages.
- Unfortunately, that theory is beyond the scope of this class.

Prandtl-Meyer Expansion Waves

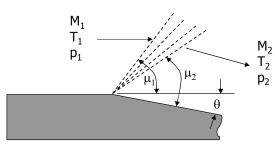
- Now let's return to the functional opposite of an oblique shock wave – the expansion fan.
- Prandtl and Meyer were the first to derive a theory for this flow, so it is often called a Prandtl-Meyer expansion.



- From experimentation, it had already been observed that expansion fans:
  - Turn the flow to be tangent to the surface
  - Accelerate the flow while T and p decrease
  - Are isentropic – total pressure and density are constant.

Prandtl-Meyer Expansion Waves [2]

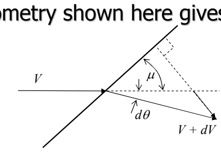
- From Schleiren photographs of the flow, it was also known that:
  - The front edge of the expansion fan is at initial Mach angle from the original flow direction.
  - The final edge of the fan is at the final Mach angle from the final flow direction.
- Thus the fan is made up of weak Mach wave disturbances which smoothly turn the flow from initial to final direction and speed.



Prandtl-Meyer Expansion Waves [3]

- Based upon this, the analysis method is to find equations for weak waves that turn the flow by $d\theta$.
- Then, integrate the equations to get the full turning angle θ.
- For a single wave, the geometry shown here gives the relations ship:

$$V\cos\mu = (V+dV)\cos(\mu + d\theta)$$



- Note that the change in velocity is normal to the wave – tangential velocity does not change.

Prandlt-Meyer Expansion Waves [4]

- The previous equation can be re-written as:

$$\frac{V+dV}{V}=\frac{\cos\mu}{\cos(\mu+d\theta)}=\frac{\cos\mu}{\cos\mu\cos d\theta-\sin\mu\sin d\theta}$$

- And, making a small angle assumption for $d\theta$, this relation becomes:

$$1+\frac{dV}{V}=\frac{\cos\mu}{\cos\mu-d\theta\sin\mu}=\frac{1}{1-d\theta\tan\mu}$$

- Or, if we apply the first term of the power series

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots$$

- This becomes: $1+\frac{dV}{V}=1+d\theta\tan\mu$

Prandlt-Meyer Expansion Waves [5]

- Finally, rearrange this and use the Mach angle definition to get: $\boxed{d\theta=\cot\mu\frac{dV}{V}=\sqrt{M^2-1}\frac{dV}{V}}$
- This is the function that will be integrated to get the total deflection:

$$\theta=\int_0^\theta d\theta=\int_{M_1}^{M_2}\sqrt{M^2-1}\frac{dV}{V}$$

- To evaluate this integral, a relationship between velocity and Mach number is needed.
- The obvious one is of course:

$$V=Ma$$

Prandlt-Meyer Expansion Waves [6]

- When differentiated this becomes:

$$dV=d(Ma)=adM+Mda$$

or

$$\frac{dV}{V}=\frac{dM}{M}+\frac{da}{a}$$

- Thus, a further relationship between Mach number and the speed of sound is needed!
- Using one form of the energy equation gives:

$$\frac{a_0^2}{a^2}=\frac{T_0}{T}=1+\frac{\gamma-1}{2}M^2$$

or

$$a=a_0\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2}$$

Prandlt-Meyer Expansion Waves [7]

- Differentiating this expression gives:

$$da = -a_0\left(\frac{\gamma-1}{2}M\right)\left(1+\frac{\gamma-1}{2}M^2\right)^{-3/2} dM$$

or

$$\frac{da}{a} = -\left(\frac{\gamma-1}{2}M^2\right)\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}\frac{dM}{M}$$

- Putting this into our earlier equation gives:

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a} = \left(1+\frac{\gamma-1}{2}M^2\right)^{-1}\frac{dM}{M}$$

- So, finally, the integral equation becomes:

$$\boxed{\theta = \int_{M_1}^{M_2}\sqrt{M^2-1}\,\frac{dV}{V} = \int_{M_1}^{M_2}\frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}}$$

Prandlt-Meyer Function

- The previous integral actually has a analytic solution – but a very long messy one.
- To simplify it's evaluation, Prandlt and Meyer introduced a new flow parameter:

$$\nu(M) = \int_{1}^{M}\frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

- This parameter, called the Prandlt-Meyer function, is the angle a flow would have turned through to get to speed, M, if it started at the speed of sound.
- The value of $\nu(M)$, in degrees, is tabulated in the supersonic tables of NACA 1135 – or in Appendix C of the textbook.

Prandlt-Meyer Function [2]

- The function can also be evaluated analytically as:

$$\boxed{\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1}}$$

- The value of this new function comes from splitting the integral for t:

$$\theta = \int_{M_1}^{M_2}\frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} = \int_{1}^{M_2}\frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} - \int_{1}^{M_1}\frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

- Thus, the turning angle is just the difference between final and initial P-M functions:

$$\theta = \nu(M_2) - \nu(M_1)$$

- Or, even better, the final P-M function (and thus M_2) can be found from the initial value and the turning angle: $\boxed{\nu(M_2) = \nu(M_1) + \theta}$

Property Changes across a P-M Expansion

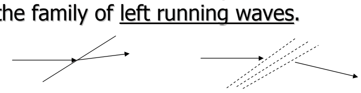
- One final note. Since the flow across an P-M expansion is isentropic, our previous isentropic flow relations can be used.
- Thus, across a P-M expansion:

$$\frac{T_2}{T_1} = \frac{T_2/T_{02}}{T_1/T_{01}} = \frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2} \qquad \frac{p_2}{p_1} = \frac{p_2/p_{02}}{p_1/p_{01}} = \left(\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right)^{\gamma/\gamma-1}$$

$$\frac{\rho_2}{\rho_1} = \frac{\rho_2/\rho_{02}}{\rho_1/\rho_{01}} = \left(\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right)^{1/\gamma-1}$$

Wave Families

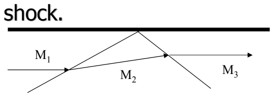
- The next topic for discussion is wave reflections and interactions.
- However, before doing that, it is useful to present a way of classifying waves into families.
- If a wave runs to the left as you follow it down, it is in the family of <u>left running waves</u>.



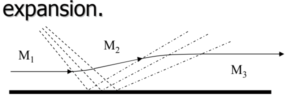
- If a wave runs to the right as you follow it down, it is in the family of <u>right running waves</u>.

Solid Wall Reflections

- <u>A wave will reflect off of a solid wall as a wave of the same type, but opposite family</u>.
- Thus a left running shock reflects as a right running shock.



  - Note: this assumes that $M_2 > 1$.
- Or, a right running expansion reflects as a left running expansion.

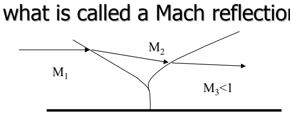


Solid Wall Reflections [2]

- Note that these are not specular (i.e. mirror like) reflections – the reflective wave is not at the same angle as the incident wave.
- Instead, the wave strength (and thus its angle) is determined by the requirement of flow tangency to the surface.
- Since the turning angles for each wave are the same magnitude then:
  - For a shock reflection, $M_1 > M_2$, therefore, $\mu_1 < \mu_2$., and the reflection is steeper then the original wave.
  - For an expansion reflection, $M_1 < M_2$, therefore, $\mu_1 > \mu_2$, and the reflection is shallower then the original.

Mach Reflection

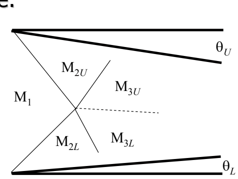
- The previous result is only valid as long as $\theta < \theta_{max}$ for the reflection shock as well as the incident one.
- For some cases, the Mach number after the first shock is so low that it can no longer turn as far.
- In this case, the incident wave turns normal to the wall in what is called a Mach reflection.



- The normal shock produces locally subsonic flow on the wall – thus this is a transonic flow problem.

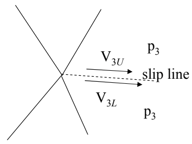
Opposite Family/Same Type Intersections

- <u>If two wave of opposite families, but the same type, intersect they reflect off each other as waves of the same type, but opposite family</u>.
- Thus, if there is an inlet with ramps on upper and lower walls, the flow looks like:
- The strength of the reflected waves is determined by two requirements in region 3:
  - The two flows must be tangent to one another.
  - The two flows must be at the same pressure!



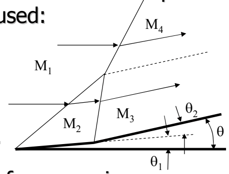
Opposite Family/Same Type [2]

- If the flow is isentropic, i.e. expansion or Mach waves, the upper and lower flows would just have to turn back horizontal.
- This is not true for shocks, however, due to the total pressure losses (entropy increases).
- Instead, the two final flows end up slightly inclined from the horizontal – and the two velocities will not be the same.
- The border between upper and lower flow – with an entropy and thus velocity jump is called a slip line.



Same Family/Same Type Intersection

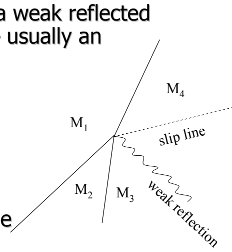
- If two wave of the same family and the same type intersect they combine to produce a stronger wave of the same type and family.
- This can be visualized as the case where a ramp with two increasing angles is used:
- The two shock waves will always converge due to both the initial ramp angle and the lower value for M_2 (higher μ_2).



- Note that there is no corollary for expansion waves since they always turn away from each other!

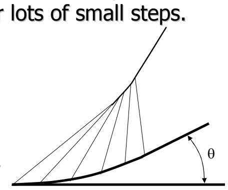
Same Family/Same Type Intersection [2]

- Where the two waves coalesce into one, the same requirements as before exist: flow tangency and equal pressures.
- To meet these requirements, a weak reflected wave comes off the junction – usually an expansion.
- Also, a slip line again forms due to the different histories of the two flow paths.
- In practice, it is usually sufficient to assume the flow above the junction turns by the total turning angle, θ.



Isentropic Compression Ramp

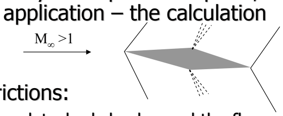
- An interesting situation arises when a ramp is used which has a smooth curve – or lots of small steps.
- The flow in this case is turned by a series of very weak compression waves.
- As a result, the flow along the surface may be isentropic – or very close to it.



- However, a shock will still form away from the surface where the Mach waves eventually coalesce.

Shock-Expansion Theory

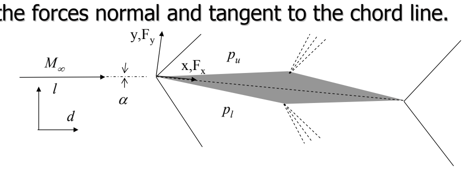
- Now that all the preliminary concepts are in place, let's turn to a practical application – the calculation of forces on an airfoil



- We will make two restrictions:
  - First, there cannot be any detached shocks, and the flow must remain supersonic.
  - The Mach number must be low enough that wave interactions away from the surface can be neglected.
- The 2-D airfoil analysis performed is called Shock-Expansion theory – because the method uses the shock jump and P-M expansion results we just learned.

Shock-Expansion Theory [2]

- To calculate the surface pressures, use the appropriate equations for the type and family of wave being generated.
- To calculate the lift and drag per unit span, the pressures must be integrated over the chord.
- To do this, it is usually easier to integrate to find the forces normal and tangent to the chord line.



Shock-Expansion Theory [3]

- The integral equations are:

$$F_y = \int_0^c (p_l - p_u)\,dx \qquad F_x = \int_0^c \left(p_u \frac{dy_u}{dx} - p_l \frac{dy_l}{dx} \right) dx$$

- And, rotating into the flow direction gives:

$$l = F_x \cos\alpha - F_y \sin\alpha \qquad d = F_x \cos\alpha + F_y \sin\alpha$$

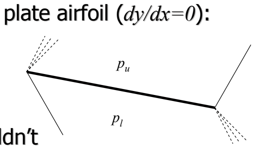
- Of course, for simple geometries, these equations become simpler also.

Shock-Expansion Theory [4]

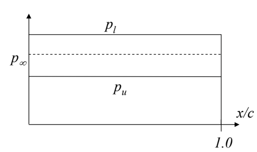
- For, example, on a flat plate airfoil (*dy/dx=0*):

$$l = \cos\alpha (p_l - p_u) c$$

$$d = \sin\alpha (p_l - p_u) c$$



- This drag – which wouldn't occur in subsonic flow is what we call **wave drag**.
- Since the pressures, and thus difference in pressures, is constant, the center of pressure is the mid chord, *x/c=0.5*.

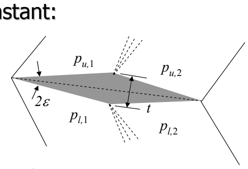


Shock-Expansion Theory [4]

- For the diamond, or double wedge, airfoil, the pressures are piecewise constant:

$$F_y = (p_{l,1} + p_{l,2} - p_{u,1} - p_{u,2})\frac{c}{2}$$

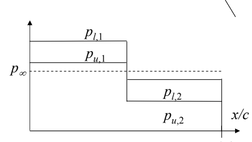
$$F_x = \tan\varepsilon (p_{l,1} + p_{u,1} - p_{l,2} - p_{u,2})\frac{c}{2}$$



- Note that the thickness is related to the wedge angle by:

$$\tan\varepsilon = \frac{t}{c}$$

- However, the center of pressure is still at the mid-chord!



Thin Airfoil Theory

- The previous method can become rather tedious for complex geometries.
- However, it is an "exact" method and must be used if there are strong shocks present.
- However, most airfoils are thin and fly at low angles of attach at supersonic Mach numbers.
- Thus, it is natural to see if there is an approximate method suitable for weak shocks and nearly isentropic flow.
- This method is called Thin Airfoil Theory.

Thin Airfoil Theory [2]

- To develop this method, start with the weak Mach expansion equation derived earlier:

$$d\theta = \sqrt{M^2 - 1}\,\frac{dV}{V}$$

- First, let's generalize this equation to also allow for weak compressions.
- To do this, we need to standardize our angles such that an compression turn is positive, an expansion turn is negative:

$$d\theta = -\sqrt{M^2 - 1}\,\frac{dV}{V}$$

- Next, use Euler's momentum equation:

$$dp = -\rho V dV$$

Thin Airfoil Theory [3]

- To get the new equation: $d\theta = \sqrt{M^2 - 1}\,\frac{dp}{\rho V^2}$
- Which can be rewritten as:

$$\frac{dp}{\frac{1}{2}\rho V^2} = \frac{2d\theta}{\sqrt{M^2 - 1}}$$

- Now, make the first assumption. Rather than integrate this expression, assume that the turning angle is small - and thus the pressure change also.
- Next, reference the turning from the initial freestream flow direction. The equation is then:

$$\frac{\Delta p}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{2\Delta\theta}{\sqrt{M_\infty^2 - 1}}$$

Thin Airfoil Theory [4]

- Or, expanding the difference terms:

$$\frac{p-p_\infty}{\frac{1}{2}\rho V_\infty^2}=\frac{2(\theta-\theta_\infty)}{\sqrt{M_\infty^2-1}}$$

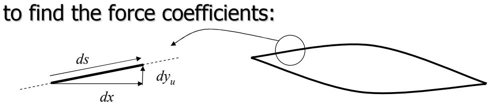
- And, finally, let the initial flow angle, θ_∞, be zero and recognize the left hand term is just the pressure coefficient:

$$\boxed{C_p=\frac{2\theta}{\sqrt{M_\infty^2-1}}}$$

- This equation relates the local pressure coefficient to the local angle from the freestream direction.
- It is an approximate relation, but can be derived from may ways – we just picked the easiest one.

Thin Airfoil Theory [5]

- This pressure coefficient can in turn be integrated to find the force coefficients:



- The contributions to lift and drag from a small segment of upper or lower skin segment are:

$$dC_l=\left(C_{p,l}\frac{dx_l}{ds}-C_{p,u}\frac{dx_u}{ds}\right)ds \qquad dC_d=\left(C_{p,u}\frac{dy_u}{ds}-C_{p,l}\frac{dy_l}{ds}\right)ds$$

- Note the signs were selected to ensure the correct result given the local surface slope.

Thin Airfoil Theory [6]

- These small contributions are integrated over the entire surface to get:

$$C_l=\frac{1}{c}\int_{LE}^{TE}\left(C_{p,l}\frac{dx_l}{ds}-C_{p,u}\frac{dx_u}{ds}\right)ds=\int_0^{\cos\alpha}\left(C_{p,l}-C_{p,u}\right)d\left(x/c\right)$$

$$C_d=\frac{1}{c}\int_{LE}^{TE}\left(C_{p,u}\frac{dy_u}{ds}-C_{p,l}\frac{dy_l}{ds}\right)ds=\int_0^{\cos\alpha}\left(C_{p,u}\frac{dy_u}{dx}-C_{p,l}\frac{dy_l}{dx}\right)d\left(x/c\right)$$

- Also note that the local surface slopes are related to the surface angles by:

$$\frac{dy_u}{dx}=\tan(\theta_u) \qquad \frac{dy_l}{dx}=\tan(\theta_l)$$

- So far, our analysis is exact – now lets make small angle approximations.

Thin Airfoil Theory [7]

- If the body is thin and does not have a blunt nose, then the surface angles are small. Thus:

$$\frac{dy_u}{dx} \approx \theta_u \qquad \frac{dy_l}{dx} \approx \theta_l$$

- And we can use the approximate form for the pressure coefficients we obtained earlier:

$$C_{p,u} = \frac{2\theta_u}{\sqrt{M_\infty^2 - 1}} \qquad C_{p,l} = \frac{-2\theta_l}{\sqrt{M_\infty^2 - 1}}$$

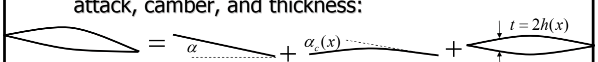
- Note the negative sign on the lower surface accounts for having waves of the opposite family.
- Also, if the angle of attack is small then $\cos\alpha \approx 1.0$

Thin Airfoil Theory [8]

- With these approximations, our lift and drag coefficient equations become:

$$C_l = \frac{-2}{\sqrt{M_\infty^2 - 1}} \int_0^1 (\theta_l + \theta_u)\, d\left(\frac{x}{c}\right) \qquad C_d = \frac{2}{\sqrt{M_\infty^2 - 1}} \int_0^1 (\theta_l^2 + \theta_u^2)\, d\left(\frac{x}{c}\right)$$

- For analysis purposes, we can decompose the surface slope into three contributions: angle of attack, camber, and thickness:



- Such that the surface angles are:

$$\theta_u = -\alpha - \alpha_c(x) + \frac{dh(x)}{dx} \qquad \theta_l = -\alpha - \alpha_c(x) - \frac{dh(x)}{dx}$$

Thin Airfoil Theory [9]

- When the upper and lower surface slopes are added for our lift equation, the thickness terms cancel to get: $C_l = \frac{4}{\sqrt{M_\infty^2 - 1}} \int_0^1 (\alpha + \alpha_c)\, d\left(\frac{x}{c}\right)$
- But, when integrated, the camber term also dissapears since the camber line starts and ends at the same height!
- Thus, the lift coefficient is: $C_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$
- Note that supersonically, lift is only generated by angle-of-attack.
- In subsonic flow, camber also contributes to lift.

Thin Airfoil Theory [10]

- The two surface slopes, squared, when added becomes:

$$\theta_l^2 + \theta_u^2 = 2\left[\alpha^2 + \alpha_c^2 + \left(\frac{dh}{dx}\right)^2 + 2\alpha\alpha_c\right]$$

- So the drag coefficient integration is:

$$C_d = \frac{4}{\sqrt{M_\infty^2 - 1}}\int_0^1\left[\alpha^2 + \alpha_c^2 + \left(\frac{dh}{dx}\right)^2 + 2\alpha\alpha_c\right]d\left(x/c\right)$$

- However, the last term again integrates to zero.
- Thus, the drag coefficient can be written as:

$$\boxed{C_d = \frac{4}{\sqrt{M_\infty^2 - 1}}\left[\alpha^2 + \overline{\alpha_c^2} + \overline{\left(\frac{dh}{dx}\right)^2}\right]}$$

Thin Airfoil Theory [11]

- Where the bar indicates an averaging:

$$\overline{\alpha_c^2} = \int_0^1 \alpha_c^2 \, d\left(x/c\right) \qquad \overline{\left(\frac{dh}{dx}\right)^2} = \int_0^1 \left(\frac{dh}{dx}\right)^2 d\left(x/c\right)$$

- Thus, camber and thickness contribute nothing to lift but significantly to wave drag.
- Thus, supersonic airfoils are as thin as we can structurally make them, and usually have no camber.
- However, if you have to also fly subsonic, then you will have to suffer the penalty of higher thickness and or camber to delay stall separation.