

## Boundary Conditions

- To solve the Boundary Layer Equations, we need to have boundary conditions. One is obvious, no-slip at the wall:

$$y = 0 \Rightarrow u = v = 0 \quad T = T_w$$

- The second is the requirement that the velocity and temperature at the edge of the boundary layer should approach the those of the undisturbed flow,  $u_e$  and  $T_e$ .
- Since the velocity at the boundary layer thickness is only 99% (or 95%, depending upon definition) that of the freestream, this boundary condition is normally expressed as a limit as  $y$  approaches infinity.

$$y \rightarrow \infty \Rightarrow u \rightarrow u_e(x) \quad T \rightarrow T_e(x)$$

- Also, the pressure gradient along the wall must be given.

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## Impact of Pressure Gradient

- Some impacts of the pressure gradient on the BL flow can be made without solving the equations.

- For example, consider the flow at the wall where, since the velocity is zero, the momentum equation becomes just (assuming constant  $\mu$ ):

$$0 = -\frac{dp}{dx} + \mu \left( \frac{\partial^2 u}{\partial y^2} \right)_w$$

- This shows that the pressure gradient and the 2<sup>nd</sup> derivative of velocity at the wall are proportional and have the same sign.
- We should also observe the the 2<sup>nd</sup> derivative must approach zero from a negative value at the far field in order to asymptotically approach  $u_e$ .

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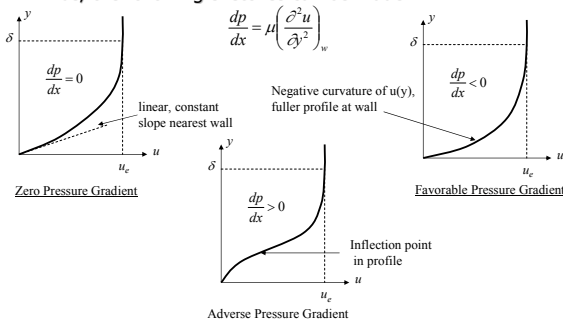
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## Impact of Pressure Gradient [2]

- Thus, the following sketches can be made:




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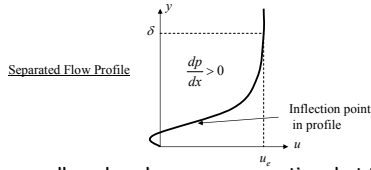
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### Impact of Pressure Gradient [3]

- This leads to another important observation – that separation only occurs in adverse pressure gradients.
- In order to have a reversed flow region, there must be a point of inflection in the BL – thus the adverse pressure.



- Sharp wall angles also cause separation, but that is because they induce large adverse pressure gradients.

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### Blassius Solution

- Let's now look at our first solution to the Boundary Layer equations for the special case of incompressible and adiabatic flow with zero pressure gradient.
- In this case, the energy equation is not needed since there is neither compressibility or heat transfer.
- The remaining BL equations can be written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2} = \nu \frac{\partial^2 u}{\partial y^2}$$

- With the boundary conditions:

$$y = 0 \Rightarrow u = v = 0$$

$$y \rightarrow \infty \Rightarrow u \rightarrow u_e = V_\infty$$

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### Blassius Solution [2]

- A solution method proposed by Blassius is based upon a similarity transformation of the equations.
- Experimentation shows that a laminar BL in these conditions has "similar" velocity profiles– i.e. the BL may grow in thickness with distance but the basic shape is always the same.
- As a result, Blassius suggested a solution of the type:

$$u(x, y) = V_\infty F(\eta) \quad \eta = \frac{y}{g(x)}$$

- The function,  $F(\eta)$ , is the solution for all  $x$ , but it depends upon the  $\eta$  which is a scaled function of  $y$ .
- The trick is to select a good scaling parameter,  $g(x)$ .

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### Blassius Solution [3]

- Experimentation has also shown that the boundary layer grew in proportion to position by:  $\frac{\delta}{x} \propto \frac{1}{\sqrt{\text{Re}_x}}$

- Thus, Blassius assumed a scaling function in the form:

$$g(x) = \frac{x}{\sqrt{\text{Re}_x}} = \sqrt{\frac{\mu x}{\rho V_\infty}} \quad \eta = \frac{y}{g(x)} = y \sqrt{\frac{\rho V_\infty}{\mu x}}$$

- Another modification which is used to simplify the analysis is the introduction of the stream function,  $\psi$ :

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}$$

- The stream function only exists in 2-D flow and has the nice feature that it automatically satisfies continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

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### Blassius Solution [4]

- Finally, lets make the conversion from our traditional coordinates  $(x,y)$  to a new set of coordinates  $(\xi,\eta)$ :

$$\xi = x \quad \eta = y \sqrt{\frac{\rho V_\infty}{\mu x}}$$

- In making this transformation, we will need to also transform our derivatives. For example:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - \frac{\eta}{2x} \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} = \sqrt{\frac{\rho V_\infty}{\mu x}} \frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} \right) = \frac{\rho V_\infty}{\mu x} \frac{\partial^2}{\partial \eta^2}$$

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### Blassius Solution [5]

- We can observe that applying these transformations to the stream function results in:

$$u = \frac{\partial \psi}{\partial y} = \sqrt{\frac{\rho V_\infty}{\mu x}} \frac{\partial \psi}{\partial \eta}$$

- However, we also have the assume functionality:

$$u = V_\infty F(\eta)$$

- Thus, the stream function is related to our velocity profile function by:

$$\frac{\partial \psi}{\partial \eta} = \sqrt{\frac{\mu x V_\infty}{\rho}} F(\eta) = \sqrt{v_\infty^2 x} F(\eta)$$

- Or alternately, introduce a new function,  $f(\eta)$ , defined by:

$$\psi(\xi, \eta) = \sqrt{v_\infty^2 x} f(\eta) \quad f'(\eta) = \frac{df}{d\eta} = F(\eta)$$

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### Blassius Solution [6]

- Finally, let's put all these definitions together to transform our remaining flow equation:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

- First the velocities are transformed:

$$u = V_\infty f'$$

$$\begin{aligned} v &= -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x} (\sqrt{\nu x} V_\infty f(\eta)) \\ &= -\frac{\partial}{\partial \xi} (\sqrt{\nu x} V_\infty f(\eta)) + \frac{\eta}{2x} \frac{\partial}{\partial \eta} (\sqrt{\nu x} V_\infty f(\eta)) \\ &= -\frac{1}{2} \sqrt{\frac{\nu V_\infty}{x}} (f - \eta f') \end{aligned}$$

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### Blassius Solution [7]

- The first derivatives of velocity are:

$$\begin{aligned} \frac{\partial u}{\partial x} &= V_\infty \frac{\partial \eta}{\partial x} f'' = -\frac{V_\infty \eta}{2\xi} f'' \\ \frac{\partial u}{\partial y} &= \sqrt{\frac{V_\infty}{\nu x}} \frac{\partial}{\partial \eta} (V_\infty f') = V_\infty \sqrt{\frac{V_\infty}{\nu x}} f'' \\ \frac{\partial^2 u}{\partial y^2} &= V_\infty \frac{V_\infty}{\nu x} f''' \end{aligned}$$

- All together, this becomes:

$$V_\infty f' \left( -\frac{V_\infty \eta}{2\xi} f'' \right) - \frac{1}{2} \sqrt{\frac{\nu V_\infty}{x}} (f - \eta f') \left( V_\infty \sqrt{\frac{V_\infty}{\nu x}} f'' \right) = \nu \left( V_\infty \frac{V_\infty}{\nu x} f''' \right)$$

- Which looks very daunting. However, after expanding the terms, this becomes.

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### Blassius Solution [8]

$$-\frac{V_\infty^2 \eta}{2\xi} f' f'' - \frac{V_\infty^2}{2\xi} f f'' + \frac{V_\infty^2 \eta}{2\xi} f' f'' = \frac{V_\infty^2}{\xi} f'''$$

- And, the 1<sup>st</sup> and 3<sup>rd</sup> terms cancel out – leaving just:

$$2f''' + ff'' = 0$$

- Which is an ordinary differential equation with boundary conditions:

$$\eta = 0 \quad \Rightarrow \quad f = f' = 0$$

$$\eta \rightarrow \infty \quad \Rightarrow \quad f' \rightarrow 1$$

- However, this is not a linear ODE! Thus, it must be solved using a numerical approach.

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## Displacement Thickness

- You may have noticed that the definition of the boundary layer thickness is rather imprecise – it also lacks much physical significance.
- There are two other thickness like properties which are both better defined and physically significant.
- The first is called the Displacement Thickness, given either the symbol  $\delta^*$  or  $\delta_1$ , depending upon the author.
- To visualize what displacement thickness means, imagine a control volume on a flat plate that is bounded by the wall on one side and a streamline in the undisturbed flow on the other.

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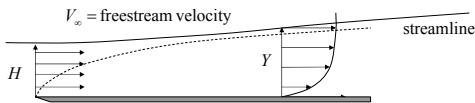
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## Displacement Thickness [2]



- To conserve mass, the downstream height of the streamline  $Y$ , is greater than the initial height,  $H$ .
- This difference in height is the displacement thickness,  $\delta^* = Y - H$ . Mathematically it can be found by applying conservation of mass to the C.V.

$$\oint \rho(\vec{V} \cdot \vec{n}) dS = - \int_0^H \rho V_\infty dy + \int_0^Y \rho u dy = 0$$

$$\rho V_\infty H = \int_0^Y \rho u dy$$

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## Displacement Thickness [3]

- Or, with manipulation:

$$\rho_\infty V_\infty H = \int_0^Y \rho u dy = \int_0^Y (\rho u - \rho_\infty V_\infty + \rho_\infty V_\infty) dy = \int_0^Y (\rho u - \rho_\infty V_\infty) dy + \rho_\infty V_\infty Y$$

$$\delta^* = Y - H = \frac{1}{\rho_\infty V_\infty} \int_0^Y (\rho_\infty V_\infty - \rho u) dy = \int_0^Y \left( 1 - \frac{\rho u}{\rho_\infty V_\infty} \right) dy$$

- Note that this integrand goes to zero outside the boundary layer – thus we can integrate to any height greater than  $\delta$ .
- This gives us our general formula for the displacement thickness.

$$\delta^* = \int_0^\infty \left( 1 - \frac{\rho u}{\rho_\infty V_\infty} \right) dy$$

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## Momentum Thickness

- The other thickness parameter of significance is called the Momentum Thickness, given the symbol  $\theta$  or  $\delta_2$ .
- While the displacement thickness can be thought of as being related to the deficit of mass flux in a BL, momentum thickness is related to the momentum flux deficit – and thus the drag force.
- The derivation is similar to before, but now conservation of momentum in the CV is used:

$$\iint \rho u(\vec{V} \cdot \vec{n}) dS = \iint \tau dS - \iint p(\vec{i} \cdot \vec{n}) dS$$

$$-\int_0^H \rho V_\infty^2 dy + \int_0^y \rho u^2 dy = -drag = -d$$

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## Momentum Thickness [2]

- Or, with simplification:  $d = \rho_\infty V_\infty^2 H - \int_0^y \rho u^2 dy$

- But, from continuity:

$$\rho_\infty V_\infty H = \int_0^y \rho u dy$$

- Thus, with substitution:

$$d = V_\infty \int_0^y \rho u dy - \int_0^y \rho u^2 dy = \int_0^y \rho u (V_\infty - u) dy$$

- The momentum thickness is then defined by:

$$\theta = \frac{d}{\rho_\infty V_\infty^2} = \int_0^\infty \frac{\rho u}{\rho_\infty V_\infty} \left(1 - \frac{u}{V_\infty}\right) dy$$

- Note the upper limit of integration was changed since this integrand also goes to zero outside the BL.

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## Blassius Solution For $\delta^*$ and $\theta$

- With these definitions, use the Blassius solution to find the two new thicknesses for incompressible flow.

- For displacement thickness, the integration results in:

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{V_\infty}\right) dy = \int_0^\infty (1 - f') \frac{dy}{d\eta} d\eta = \sqrt{\frac{15x}{V_\infty}} \int_0^\infty (1 - f') d\eta = \sqrt{\frac{15x}{V_\infty}} (\eta - f)$$

- However, both  $\eta$  and  $f$  are zero at the wall, and their difference approaches the fixed value of  $\sim 1.72$  far away from the wall.

- Thus:

$$\delta^* = 1.72 \sqrt{\frac{15x}{U_\infty}}$$

$$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\text{Re}_x}}$$

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### Blassius Solution For $\delta^*$ and $\theta$ [2]

- For momentum thickness, the integration results in:

$$\theta = \int_0^{\infty} \frac{u}{V_{\infty}} \left(1 - \frac{u}{V_{\infty}}\right) dy = \sqrt{\frac{12x}{V_{\infty}}} \int_0^{\infty} f'(1-f') d\eta$$

- Unfortunately, this does not have a simple solution. However, numeric integration of the Blassius solution gives:

$$\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$

- A result which could have been anticipated given the previous solutions for skin friction and drag coefficients.

- Namely that:

$$C_d = 2 \frac{\theta_{x=c}}{c}$$

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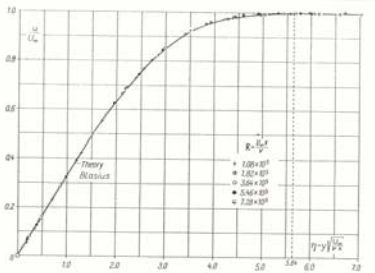
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### Accuracy of the Blassius Solution

- The question remains as to how accurate is the Blassius solution – and the comparison below with experimental measurements show it is very good.




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