Introduction to Turbulence

- By these point in your studies, you have probably heard a fair amount about what turbulence is and how it effects the flow in pipes or on bodies.
- However, you have probably not seen very much theoretical modeling of turbulence – mainly because it is a complex and evolving field without simple solutions.
- What we will attempt to do in this section is to present the theoretical basis for the turbulence modeling of incompressible fluids – without getting into the details.
- Hopefully, this introduction will help you in understanding the issues involved and impact of turbulence better.

Introduction to Turbulence [2]

- Turbulence is usually characterized by unsteady, chaotic but motion with an underlying well define average.
- Thus, the flow variables, and in particular velocity, are functions of time as well as position:

$$u(x,y,t) \qquad v(x,y,t)$$

- However, because of the existence of an average velocity we can write these velocities as the sum of a steady, time invariant component and a fluctuation one:

$$u(x,y,t)=\bar{u}(x,y)+u'(x,y,t) \qquad v(x,y,t)=\bar{v}(x,y)+v'(x,y,t)$$

- Where the average or mean values are defined by the time average:

$$\bar{u}(x,y)=\frac{1}{T}\int_0^T u(x,y,t)dt \qquad \bar{v}(x,y)=\frac{1}{T}\int_0^T v(x,y,t)dt$$

Introduction to Turbulence [3]

- Where the length of time used for averaging, T, need only be enough to obtain a good steady mean value.
- The fluctuating velocities are then define by the difference:

$$u'(x,y,t)=u(x,y,t)-\bar{u}(x,y) \qquad v'(x,y,t)=v(x,y,t)-\bar{v}(x,y)$$

- Note that by definition, the time average of the fluctuating velocities is zero:

$$\frac{1}{T}\int_0^T u'(x,y,t)dt=0 \qquad \frac{1}{T}\int_0^T v'(x,y,t)dt=0$$

- Where this idea of splitting the properties into mean and fluctuations is useful is in considering the time averages of our conservation equations.

Reynolds Equations

- Consider our 2-D, incompressible boundary layer equations that we used for the Blassius solution.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

- To simplify the following process, let's rewrite the momentum equation by adding in the continuity equation multiplied by the horizontal velocity, u.

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

- Or, noticing that some of the terms on the left-hand-side can be combined using reverse differentiation by parts…

$$\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

Reynolds Equations [2]

- Now, replace the instantaneous velocities with the split sum of mean and fluctuation values:

$$u=\overline{u}+u' \qquad v=\overline{v}+v'$$

- To give the rather lengthy result:

$$\frac{\partial \overline{u}}{\partial x}+\frac{\partial u'}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial v'}{\partial y}=0$$

$$\frac{\partial \overline{u}\,\overline{u}}{\partial x}+2\frac{\partial \overline{u}u'}{\partial x}+\frac{\partial u'u'}{\partial x}+\frac{\partial \overline{u}\,\overline{v}}{\partial y}+\frac{\partial u'\overline{v}}{\partial y}+\frac{\partial \overline{u}v'}{\partial y}+\frac{\partial u'v'}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 \overline{u}}{\partial y^2}+\frac{\mu}{\rho}\frac{\partial^2 u'}{\partial y^2}$$

- This hardly seems like an improvement – but now consider time averaging the entire equations.
- Any term which involves a product of a fluctuating quantity alone or in product with a mean value vanishes.

Reynolds Equations [3]

- Only terms with only mean values or products of fluctuating values will remain.

$$\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}=0$$

$$\frac{\partial \overline{u}\,\overline{u}}{\partial x}+\frac{\partial \overline{u}\,\overline{v}}{\partial y}+\frac{1}{T}\int_0^T\frac{\partial u'u'}{\partial x}dt+\frac{1}{T}\int_0^T\frac{\partial u'v'}{\partial y}dt=\frac{\mu}{\rho}\frac{\partial^2 \overline{u}}{\partial y^2}$$

- These two new terms can be written in simplified from as: $\overline{u'u'}=\frac{1}{T}\int_0^T u'u'dt \qquad \overline{u'v'}=\frac{1}{T}\int_0^T u'v'dt$
- Thus, the time averaged, or Reynolds averaged momentum equation is:

$$\frac{\partial \overline{u}\,\overline{u}}{\partial x}+\frac{\partial \overline{u}\,\overline{v}}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 \overline{u}}{\partial y^2}-\frac{\partial \overline{u'u'}}{\partial x}-\frac{\partial \overline{u'v'}}{\partial y}$$

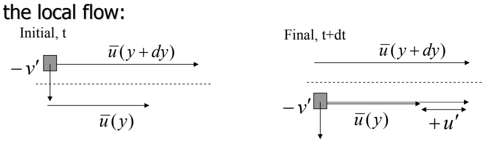
Reynolds Equations [4]

- You might wonder why these two new terms do not drop out in the averaging process.
- The first term does not average to zero because the product of a fluctuation with itself must be a positive.
- Since the integrand can only be positive (or zero), the integration must also yield a positive result.
- However, this term, which represents a turbulent axial stress in the flow, is usually quite small and is neglected in normal boundary layer analysis.

$$\overline{u'u'} \approx 0$$

Reynolds Equations [5]

- On the other hand, the term with the product of the x and y velocity fluctuations would have a zero time average if the fluctuations in the two axis were uncorrelated – i.e. they had no relation to each other.
- But, in a shear layer, there is a correlation!
- A chunk or packet of fluid in one shear layer which has a downward fluctuations, negative *v'*, arrives at a lower layer with a positive fluctuation, positive *u'*, relative to the local flow:



Reynolds Equations [6]

- The reverse occurs for un upward fluctuation, positive *v'* but negative *u'*.
- Thus, the integrand will more often be a negative number in a shear layer – and the integration will yield a negative value. $\overline{u'v'} \le 0$
- This term is called the turbulent shear stress, or apparent turbulent stress, or simply the Reynolds stress.
- This is the additional shear stress we associate with turbulent flow and it can be many orders of magnitude greater than the laminar shear stress.

Reynolds Equations [7]

- Using the previous discussions then, we arrive at the time averaged conservation equation often called the Reynolds equations:

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \qquad \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \overline{u}}{\partial y} - \rho\overline{u'v'}\right)$$

- These equations, or the similar equations in 3-D compressible flow, form the basis of the great majority of turbulent flow analysis with fairly good success.
- However, a lot of modern research in turbulence is based upon the modeling the unsteady rather than time-averaged equations.
- Apparently, much behavior current models don't predict well may be due to the very unsteadiness itself.

Turbulence Modeling

- The impact of turbulence my be thought of as producing an additional, turbulent viscosity similar to its laminar counterpart. I.e.:

$$\tau_{xy} = (\mu + \mu_t)\frac{\partial \overline{u}}{\partial y}$$

- This assumption is knows as the Boussinesq analogy.
- From the turbulent shear stress equation just derived, it follows that the turbulent stress and viscosity would be:

$$\tau_{xy,turb} = -\rho\overline{u'v'} \qquad \mu_t = -\frac{\rho\overline{u'v'}}{\partial \overline{u}/\partial y}$$

- The field of study called Turbulence Modeling is essentially trying to develop a mathematical model for the above terms that is accurate – and hopefully not to difficult to evaluate.

Turbulence Modeling [2]

- The difficulty in turbulence modeling is that the turbulent viscosity is a flow property, not a fluid property.
- Thus, a good turbulent model would depend upon:
  - Local flow velocities and velocity gradients.
  - The history of the flow before the local time and location.
  - The effects of surface roughness and surface geometry.
- Unfortunately, to do all of these, the model will probably not be simply and easy to evaluate.
- We will look at two relatively simple models:
  - Prandlt's Mixing Length Model
  - The Baldwin-Lomax Model

Prandlt's Mixing Length Model

- Prandlt originally proposed the concept that the velocity perturbations were due to turbulent eddies in the flow.
- As a result, the magnitude of the perturbations should depend upon the characteristic size, or length, of the eddies and the gradient in mean velocity. i.e

$$u' \propto \left(l_1 \frac{\partial \bar{u}}{\partial y} \right) \qquad v' \propto \left(l_2 \frac{\partial \bar{u}}{\partial y} \right)$$

- Thus, the Reynolds stress would be:

$$-\rho \overline{u'v'} \propto \rho \left(l_1 \frac{\partial \bar{u}}{\partial y} \right) \left(l_2 \frac{\partial \bar{u}}{\partial y} \right)$$

- Or the turbulent viscosity could be written as:

$$\mu_T = \rho l^2 \left| \frac{\partial \bar{u}}{\partial y} \right|$$

Prandlt's Mixing Length Model [2]

- The absolute value is because the viscosity should always be positive no matter the sign of the gradient.
- The task is then to determine reasonable values for the characteristic mixing length, l.
- Two important observations help in this regard.
- The first is that the mixing length must go to zero at the wall itself since there cannot be flow through the wall.
- This leads to the idea of a laminar sub-layer – a region nearest to the wall were there is only laminar stress.
- The second observation is that the mixing length in the outer boundary layer approaches a constant value that is some fraction of the boundary layer thickness.

Prandlt's Mixing Length Model [3]

- Thus, one mixing length model proposed by van Driest has an inner mixing length given by:

$$l_{inner} = \kappa y \left[1 - \exp\left(\frac{-y^+}{A^+} \right) \right] \qquad y^+ = \frac{y\sqrt{\rho \tau_w}}{\mu}$$

- Where κ and A are two constant which must be specified to correlate with experiment – usually κ=0.41 and A^+=26.
- For the outer regions of the boundary layer, the mixing length is assume a constant fraction of the thickness.

$$l_{outer} = C\delta$$

- Where this new constant is usually taken as: C=0.09.
- The model switches from the inner to outer mixing lengths at the height when: $l_{inner} > l_{outer}$

Baldwin-Lomax Model

- The Prandlt Mixing Length Model with the given values for inner and outer mixing length gives an excellent prediction of the shape of a turbulent velocity profile.
- Unfortunately, it doesn't always give a great correlation to experimental measurements.
- A relatively simple and popular modification which improves the correlation is due to Baldwin and Lomax.
- They kept the inner mixing length correlation pretty much intact:

$$\mu_{T,inner} = \rho l_{inner}^2 \left| \frac{d\bar{u}}{dy} \right|$$

- With van Driest's equation for the inner mixing length.

Baldwin-Lomax Model [2]

- For the outer layer, this model uses a more complex model: $\mu_{T,outer} = \rho K C_{cp} F_{wake} F_{Kleb}$
- K and C_{cp} are two constants that will be given shortly.
- The function F_{wake} is selected to find the maximum mixing length that occurs in the BL – its form is:

$$F_{wake} = \max\left[y_{\max} F_{\max}, y_{\max} C_{wk} (u^2 + v^2) / F_{\max} \right]$$

$$F_{\max} = \max\left[y \left| \frac{\partial \bar{u}}{\partial y} \right| \left(1 - e^{-y^+/A^+} \right) \right]$$

- Where y_{max} is the height above the wall where F_{max} is evaluated – and a new constant C_{wk} has been introduced.

Baldwin-Lomax Model [3]

- The other term, known as the Klebanoff intermittency factor accounts for the fact the mixing drops off at the edge of the BL:

$$F_{Kleb} = \left[1 + 5.5 \left(C_{kleb} \frac{y}{y_{\max}} \right)^6 \right]^{-1}$$

- As you can see, more complex models attempt to correctly account for more and more of the observed features of turbulent flow.
- However, the constants which show up don't fall out of the sky – they are usually chosen to give the best correlation to experiment.
- Common selections for these constants are:

$K = 0.0168 \quad C_{cp} = 1.6 \quad C_{wk} = 0.25 \quad C_{kleb} = 0.3$

Turbulent Conductivity

- In addition to the turbulent viscosity, in high speed flows or problems with heat transfer, a turbulent conductivity is needed.
- While a separate model could be created for this factor, most researchers take the easy route and relate the viscosity and conductivity through the Prandlt number.

$$\Pr_T = \frac{c_p \mu_T}{k_T} \approx 1$$

- Where experiment has shown that the turbulent Prandlt number is very close to 1.0.

Turbulent Flat Plate Flow

- Given these turbulence models, I would love to show you some typical solutions.
- However, even the simplest solution in turbulent flow – that for a flat plate – is very computationally intensive.
- Instead, I will just repeat the time honored flat plate experimental results you have already seen:

$$\delta_T = \frac{0.37x}{\mathrm{Re}_x^{0.2}} \qquad C_{d,T} = \frac{0.074}{\mathrm{Re}_c^{0.2}}$$

- Finishing with this result is very appropriate since the whole point of turbulence modeling is to find analytic formulations which will agree with the above.