Summary of Thin Airfoil Theory

- Let review the method for Thin Airfoil Theory
  - First, split our flow into three components for freestream flow, thickness effects, and camber effects, respectively.
    $$\phi = \phi_\infty + \phi_T + \phi_C$$
  - We will require our thickness solution to satisfy the thickness BC.
    $$v_T(x,0\pm) = \pm\tfrac{1}{2}V_\infty T'$$
  - We will require the camber solution to satisfy airfoil camber BC and/or angle-of-attack effects.
    $$v_C(x,0\pm) = V_\infty(\bar{Y}' - \alpha)$$
  - The thickness solution may be represented by a line source:
    $$v_T(x,0\pm) = \pm\tfrac{1}{2}q(x) = \pm\tfrac{1}{2}V_\infty T'$$
    $$u_T(x,0\pm) = \frac{V_\infty}{2\pi}\, P\!\!\int_0^c \frac{T'(t)}{(x-t)}\,dt$$

Summary of Thin Airfoil Theory [2]

- The camber solution may be represented by a line vortex:
  $$u_C(x,0\pm) = \pm\tfrac{1}{2}\gamma(x)$$
  $$v_C(x,0\pm) = V_\infty(\bar{Y}' - \alpha) = \frac{-1}{2\pi}\, P\!\!\int_0^c \frac{\gamma(t)}{(x-t)}\,dt$$
- Knowing the thickness and camber functions, T' and $\bar{Y}'$, we should then be able, in general, to find the source and vortex line strengths.
- With these, and in particular the vortex line strength, we will find the net force (lift) on the surface by:
  $$L' = \rho V_\infty \int_0^c \gamma(x)dx = \rho V_\infty \Gamma$$
- There is also an inherit assumption the the thickness and camber can be represented by nice functions – often not the case.

Symmetric Airfoil at Angle of Attack

- Let's start with a fairly simple lift problem, that of a symmetric airfoil at angle of attack:
  $$Y_u(x) = -Y_l(x) \qquad \bar{Y}(x) = 0$$
- If the airfoil shape is given by an analytic function like that in Example 2 for symmetric airfoils, then we can find the thickness solution.
- The camber solution for vortex strength, in the absence of camber becomes:
  $$2\pi V_\infty \alpha = P\!\!\int_0^c \frac{\gamma(t)}{(x-t)}\,dt$$
- While it is possible to solve this integral (not easy), we can obtain solution by comparison with Appendix A.

Symmetric Airfoil at Angle of Attack [2]

- From Appendix A, the general solution of Cauchy's Principal Value integral of order n is:

$$I_n(\theta_0) = \int_0^{\pi} \!\!\!\!\!\! P \;\; \frac{\cos n\theta}{\cos\theta - \cos\theta_0} d\theta = \pi \frac{\sin n\theta}{\sin\theta_0}$$

- We can get our integral in this form by making the substitutions:

$$x = \frac{c}{2}(1-\cos\theta_0) \qquad t = \frac{c}{2}(1-\cos\theta) \qquad dt = \frac{c}{2}\sin\theta d\theta$$

- With these, our integral becomes:

$$2\pi V_\infty \alpha = \int_0^{\pi} \!\!\!\!\!\! P \;\; \frac{\gamma(\theta)\sin\theta}{\cos\theta - \cos\theta_0} d\theta$$

Symmetric Airfoil at Angle of Attack [3]

- By observation if the vortex strength was:

$$\gamma(\theta) = 2V_\infty \alpha \frac{\cos\theta}{\sin\theta}$$

- Then the integral is of n=1 order, and we have a solution.
- Sounds great! However, if we had a distribution like:

$$\gamma(\theta) = \frac{k}{\sin\theta}$$

- With k an arbitrary constant, we would also an integral of order n=0 – which just equals zero.
- Thus, by superposition, our possible solutions are:

$$\gamma(\theta) = 2V_\infty \alpha \frac{\cos\theta}{\sin\theta} + \frac{k}{\sin\theta}$$

Symmetric Airfoil at Angle of Attack [4]

- The net circulation (and thus force) which acts on this airfoil is found by integrating this vortex over its length:

$$\Gamma = \int_0^c \gamma(\theta) dt = \frac{c}{2}\int_0^{\pi} \gamma(\theta)\sin\theta d\theta$$
$$= \frac{c}{2}\int_0^{\pi} \left(2V_\infty \alpha \cos\theta + k\right) d\theta$$

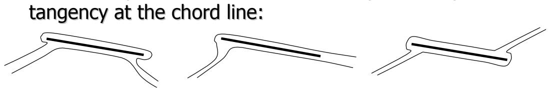
- Which gives the disconcerting result that:

$$\Gamma = \frac{c\pi k}{2}$$

- Thus, our circulation and force are going to be determined by an arbitrary constant?
- What gives?

Kutta Condition

- What is happening here is that there are an infinite number of possible solutions of vorticity to satisfy flow tangency at the chord line:



- However, being experienced aerodynamicists, we know which one we want – that with the streamlines leaving parallel at the trailing edge.
- Mathematically, we can impose this by requiring that the vortex strength at the trailing edge be zero.

$$\gamma(c) = 0$$

- This requirement is known as the Kutta Condition.

Kutta Condition [2]

- Here is a physical explanation.
  - Sharp corners have high pressure gradients, usually favorable going into a convex corner and unfavorable going out.
  - Thus, while a real (I.e. viscous) flow might be able to whip around the leading edge when the boundary layer is young…
  - It isn't going to be able to make it around the trailing edge.
  - Thus the flow separates and leaves the surface at the body angle.
- Here is a another explanation - Nature abhors a discontinuity.
  - If there was anything but a zero vorticity a the trailing edge, there would also be a pressure difference between upper and lower surface.
  - To keep this from happening, vorticity must be zero there.

Kutta Condition [3]

- Either way you explain it, the result is the same.
- Mathematically, this requirement at $x=c$ ($\theta=\pi$) is:

$$\gamma(\pi) = 2V_\infty \alpha \frac{\cos\pi}{\sin\pi} + \frac{k}{\sin\pi} = 0$$

- Or

$$k = 2V_\infty \alpha$$

- Thus, the vorticity distribution for this case is simply:

$$\gamma(\theta) = 2V_\infty \alpha \frac{1+\cos\theta}{\sin\theta}$$

- More importantly:

$$\Gamma = c\pi V_\infty \alpha \qquad L' = \rho V_\infty \Gamma = \rho V_\infty^2 c\pi\alpha$$

Lifting Symmetric Airfoils

- The lift coefficient for this case is thus:

$$\boxed{c_l = \frac{L'}{\frac{1}{2}\rho V_\infty^2 c} = 2\pi\alpha}$$

- Something you hopefully remember from Aero I.
- Considering the moment about the leading edge:

$$M'_{le} = -\rho V_\infty \int_0^c x\gamma(x)\,dx = -\frac{c^2 \rho V_\infty^2 \alpha}{2}\int_0^\pi \left(\frac{1+\cos\theta}{\sin\theta}\right)(1-\cos\theta)\sin\theta d\theta$$
$$= -\frac{c^2 \rho V_\infty^2 \alpha}{2}\int_0^\pi \sin^2\theta d\theta = -\frac{c^2 \rho V_\infty^2 \pi\alpha}{4} = -\frac{c}{4}L'$$

- Thus, the center of pressure is the quarter chord:

$$\boxed{x_{c.p.} = \frac{-M'_{le}}{L'} = \frac{c}{4}}$$

Lifting Symmetric Airfoils [2]

- Notice we derived there results without ever specifying the body shape or solving for thickness effects.
- As a result, this solution is valid for any shape as long as it has a sharp trailing edge for the Kutta condition.
- And, the solution is valid for any thickness - from a flat plate up to a 20% thick or higher.
- Practically, this solution breaks down for thick airfoils, not because the theory is bad, but thick airfoils have thick viscous layers which cannot be ignored.
- For thin airfoils, experience tells us this result for inviscid flow is accurate up till near separation angles.

Cambered Airfoils

- Now lets go back and pick up the case of a cambered airfoil.
- In this case, the solution for vorticity must satisfy:

$$2\pi V_\infty\left(\alpha - \overline{Y}'\right) = \int_0^c\!\!\!\!\!\!P\;\frac{\gamma(t)}{(x-t)}\,dt = \int_0^\pi\!\!\!\!\!\!P\;\frac{\gamma(\theta)\sin\theta}{\cos\theta - \cos\theta_0}\,d\theta$$

- Solutions which satisfy this relation can be found by trial and error - if you are very patient.
- However, experience has shown that a good assumption for vorticity is in the form:

$$\gamma(\theta)\sin\theta = 2V_\infty\left[A_0(1+\cos\theta) + \sum_{n=1} A_n \sin n\theta \sin\theta\right]$$

- Where the A_n's are constant coefficients yet to be determined.

Cambered Airfoils [2]

- The first term in this expression follows directly from the result we had for symmetric airfoils - but with an unknown constant A_0.
- The second term - or rather the sum of terms - follows from a knowledge of Cauchy integrals. Namely that:

$$P\int_0^\pi \frac{\sin n\theta \sin\theta}{\cos\theta - \cos\theta_0} d\theta = -\pi \cos n\theta_0$$

- This this assume form for vorticity, we get:

$$\alpha - \overline{Y}' = \frac{1}{\pi} P\int_0^\pi \frac{(1+\cos\theta) + \sum A_n \sin n\theta \sin\theta}{\cos\theta - \cos\theta_0} d\theta$$
$$= A_0 - \sum_{n=1} A_n \cos n\theta_0$$

- Which looks like an infinite cosine Fourier series!

Cambered Airfoils [3]

- Take a quick reality check…
- What we are after is a solution for vorticity which we assumed will have the form:

$$\gamma(\theta)\sin\theta = 2V_\infty\left[A_0(1+\cos\theta) + \sum_{n=1} A_n \sin n\theta \sin\theta\right]$$

- To find these coefficients, we will apply our boundary condition which is now in the form:

$$\alpha - \overline{Y}' = A_0 - \sum_{n=1} A_n \cos n\theta_0$$

- And finally, while this is an infinite sum, in reality we won't need more than the few terms.
- In fact, we are soon to show that only the terms up to n=2 are needed to get forces and moments.

Cambered Airfoils [4]

- But how can we evaluate any of the coefficients?
- The standard procedure used in Fourier series analysis is to make use of the orthogonality of trig functions.
- In particular, it is well know that:

$$\int_0^\pi \cos n\theta \cos m\theta d\theta = \begin{cases} 0 & m \neq n \\ \pi/2 & m = n \end{cases}$$

- Thus, we can take our boundary condition, multiply it by $\cos m\theta_0$, and integrate over the interval 0 to π.

$$\int_0^\pi (\alpha - \overline{Y}')\cos m\theta_0 d\theta_0 = \int_0^\pi \left(A_0 - \sum_{n=1} A_n \cos n\theta_0\right)\cos m\theta_0 d\theta_0$$

- All the terms on the right hand side vanish except for when m=n.

Cambered Airfoils [5]

- Thus, for the special case of n=m=0 we get:

$$A_0 = \frac{1}{\pi}\int_0^\pi \left(\alpha - \overline{Y}'\right) d\theta_0 = \alpha - \frac{1}{\pi}\int_0^\pi \overline{Y}'(x)\, d\theta_0$$

- And for the remaining cases where n=m>0 we get:

$$A_n = \frac{-2}{\pi}\int_0^\pi \left(\alpha - \overline{Y}'\right)\cos n\theta_0\, d\theta_0 = \frac{2}{\pi}\int_0^\pi \overline{Y}'(x)\cos n\theta_0\, d\theta_0$$

- While these integrations might not be easy - particularly when camber is a complex function - they are do-able.
- In performing this integrals, also note that the camber line slope must be expressed in terms of the right variable.

Forces on Cambered Airfoils

- What we are really interested in are the forces on the airfoil.
- To get the lift, or lift coefficient, we would integrate:

$$c_l = \frac{L'}{\frac{1}{2}\rho V_\infty^2 c} = \frac{2}{V_\infty c}\int_0^c \gamma(x)\,dx = \frac{1}{V_\infty}\int_0^\pi \gamma(\theta_0)\sin\theta_0\, d\theta_0$$

- With our form for vorticity this becomes:

$$c_l = 2\int_0^\pi A_0\left(1+\cos\theta_0\right)d\theta_0 + 2\int_0^\pi \sum_{n=1} A_n \sin n\theta_0 \sin\theta_0\, d\theta_0$$

- Or simply:

$$\boxed{c_l = 2\pi\left(A_0 + \tfrac{1}{2}A_1\right)}$$

- Thus, only the first series coefficient is needed for lift.

Forces on Cambered Airfoils [2]

- For moment coefficient about the leading edge:

$$c_{m,le} = \frac{M'_{le}}{\frac{1}{2}\rho V_\infty^2 c^2} = \frac{-2}{V_\infty c^2}\int_0^c x\gamma(x)\,dx = \frac{-1}{2V_\infty}\int_0^\pi \gamma(\theta_0)\left(1-\cos\theta_0\right)\sin\theta_0\, d\theta_0$$

- Which after substituting in our vorticity expression is:

$$c_{m,le} = -\int_0^\pi \left[A_0\left(1-\cos^2\theta_0\right)\right]d\theta_0 - \int_0^\pi \sum_n \left[A_n \sin n\theta_0 \sin\theta_0\right]d\theta_0 + \int_0^\pi \sum_n \left[A_n \sin n\theta_0 \sin\theta_0 \cos\theta_0\right]d\theta_0$$

- The last looks difficult, but can be simplied using the trig identity:

$$\sin\theta_0\cos\theta_0 = \tfrac{1}{2}\sin 2\theta_0$$

Forces on Cambered Airfoils [3]

- The result is then:

$$\boxed{c_{m,le} = -\frac{\pi}{2}\left(A_0 + A_1 - \tfrac{1}{2}A_2\right)}$$

- And, last but not least, the center of pressure is:

$$\boxed{x_{c.p.} = \frac{-c c_{m,le}}{c_l} = \frac{c}{4}\left(\frac{A_0 + A_1 - \frac{1}{2}A_2}{A_0 + \frac{1}{2}A_1}\right)}$$

- Thus we see that it is not necessary to find any but the first 3 coefficients.

Zero Lift Angle-of-Attack

- The equations for lift we just saw can be re-written by substituting in our expressions for the coefficients:

$$c_l = 2\pi\left(A_0 + \tfrac{1}{2}A_1\right) = 2\pi\left[\alpha - \frac{1}{\pi}\int_0^\pi \overline{Y}'(x)d\theta_0 + \frac{1}{\pi}\int_0^\pi \overline{Y}'(x)\cos\theta_0 d\theta_0\right]$$

- Or just:

$$c_l = 2\pi\left[\alpha - \frac{1}{\pi}\int_0^\pi \overline{Y}'(x)(1-\cos\theta_0)d\theta_0\right]$$

- The second term in the brackets looks like it should be an angle.
- In fact, it is the angle at which the airfoil has zero lift:

$$\boxed{\alpha_{L0} = \frac{1}{\pi}\int_0^\pi \overline{Y}'(x)(1-\cos\theta_0)d\theta_0}$$

Zero Lift Angle-of-Attack [2]

- Thus we can write the lift coefficient expression as:

$$\boxed{c_l = 2\pi[\alpha - \alpha_{L0}]}$$

- Note that for a positively camber, the integration yields a negative value for zero lift angle of attack.
- Note also that our lift curve slope has not changed. I.e.:

$$\boxed{m_0 = \frac{dc_l}{d\alpha} = 2\pi}$$

- Thus, the effect of camber is to shift our lift curve up, but not change its slope.
- One final note, the difference in angles shown above is usually called the airfoil absolute angle of attack:

$$\boxed{\alpha_a = \alpha - \alpha_{L0}}$$