

Heat Conduction

- In the intro to Heat Transfer, the conductive heat transfer was presented as:

$$q_x = -kA \frac{\partial T}{\partial x}$$

- However, this is true only in one dimension. In general, heat flux is a vector having 3 components.
- A more general equation for heat transfer is in terms of the gradient of temperature. In Cartesian coordinates:

$$\vec{q} = -kA \nabla T = \vec{i}q_x + \vec{j}q_y + \vec{k}q_z$$

$$q_x = -kA \frac{\partial T}{\partial x} \quad q_y = -kA \frac{\partial T}{\partial y} \quad q_z = -kA \frac{\partial T}{\partial z}$$

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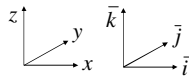
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Heat Conduction (cont.)

- Since we will be dealing with other coordinates also, realize that the gradient takes on other forms:

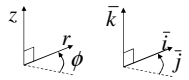
Cartesian

$$\nabla T = \vec{i} \frac{\partial T}{\partial x} + \vec{j} \frac{\partial T}{\partial y} + \vec{k} \frac{\partial T}{\partial z}$$



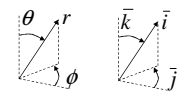
Cylindrical

$$\nabla T = \vec{i} \frac{\partial T}{\partial r} + \vec{j} \frac{1}{r} \frac{\partial T}{\partial \phi} + \vec{k} \frac{\partial T}{\partial z}$$



Spherical

$$\nabla T = \vec{i} \frac{\partial T}{\partial r} + \vec{j} \frac{1}{r} \frac{\partial T}{\partial \theta} + \vec{k} \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi}$$



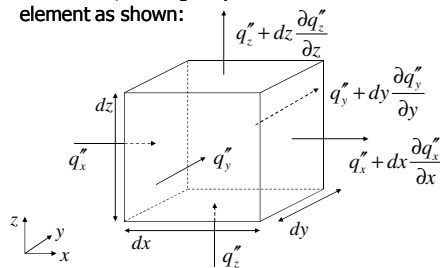
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Heat Diffusion Equation

- To develop a generalized governing law for heat conduction, we begin by considering the fluxes in an element as shown:



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Heat Diffusion Equation (cont.)

- A statement of energy conservation for this control mass (similar to the 1st Law) would be:

$$\left[\begin{array}{l} \text{Rate of change} \\ \text{of energy in the} \\ \text{element} \end{array} \right] = \left[\begin{array}{l} \text{Rate of flux} \\ \text{of energy into} \\ \text{the element} \end{array} \right] + \left[\begin{array}{l} \text{Rate of energy} \\ \text{production in} \\ \text{the element} \end{array} \right]$$

- Lets consider this statement term by term. First, the rate of change of energy:

$$\left[\begin{array}{l} \text{Rate of change} \\ \text{of energy in the} \\ \text{element} \end{array} \right] = \rho c_p \frac{\partial T}{\partial t} dx dy dz$$

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Heat Diffusion Equation (cont.)

- Note that this equation uses $\delta Q = mc_p dT$ and thus assumes constant pressure processes.

- Next consider the heat generation term:

$$\left[\begin{array}{l} \text{Rate of energy} \\ \text{production in} \\ \text{the element} \end{array} \right] = \dot{q} dx dy dz$$

$\dot{q} \equiv$ heat generation per unit volume
(W/m³)

- Sources of heat generation will be: electrical resistance, chemical reactions, nuclear reactions, or even radiation absorption like in a microwave.

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Heat Diffusion Equation (cont.)

- Lastly, consider the heat flux through the element:

$$\left[\begin{array}{l} \text{Rate of flux} \\ \text{of energy into} \\ \text{the element} \end{array} \right] = [\text{Heat influx}] - [\text{Heat outflux}]$$

$$= [q_x'' dy dz + q_y'' dx dz + q_z'' dx dy] -$$

$$\left[\left(q_x'' + \frac{\partial q_x''}{\partial x} dx \right) dy dz + \left(q_y'' + \frac{\partial q_y''}{\partial y} dy \right) dx dz + \left(q_z'' + \frac{\partial q_z''}{\partial z} dz \right) dx dy \right]$$

$$= - \left(\frac{\partial q_x''}{\partial x} + \frac{\partial q_y''}{\partial y} + \frac{\partial q_z''}{\partial z} \right) dx dy dz$$

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Heat Diffusion Equation (cont.)

- However, using the definition of q_x , q_y , and q_z this can be rewritten as:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = \frac{\partial}{\partial x} \left(-k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(-k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(-k \frac{\partial T}{\partial z} \right) = \nabla \cdot (-k \nabla T)$$

- Finally, putting all these together, we get the heat diffusion equation:

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + \dot{q}$$

- Note that this is a 2nd order differential equation!

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Heat Diffusion Equation (cont.)

- One common simplification of this equation is to assume that the material conductivity, k , is independent of position. This is valid if:
 - objects are made of a single material
 - k is not a strong function of T or T is nearly constant
- In this case, the diffusion equation can be written as:

- where $\frac{1}{\alpha} \frac{\partial T}{\partial t} = \nabla^2 T + \frac{\dot{q}}{k}$

$$\alpha = \frac{k}{\rho c_p} \equiv \text{thermal diffusivity (m}^2/\text{sec)}$$

$$\nabla^2 = \nabla \cdot \nabla \equiv \text{Laplace differential operation}$$

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Heat Diffusion Equation (cont.)

- In the three coordinate systems of interest, the Laplace operator takes the form:

Cartesian

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Cylindrical

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

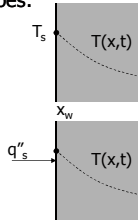
Spherical

$$\nabla^2 T = \frac{1}{r} \frac{\partial^2}{\partial r^2} (rT) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2}$$

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Heat Transfer Boundary Conditions

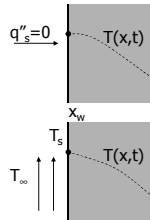
- Since heat conduction problems involve differential equations, we need to consider the different types of boundary conditions.
- In general, BC's will be one of 4 types:
 - Fixed temperature:
 - $T(x_w, t) = T_s$
 - Fixed heat flux
 - $-k(dT/dx)_{x_w} = q''_s$



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Heat Transfer BC's (cont)

- Adiabatic Wall
 - $-k(dT/dx)_{x_w} = 0$
- Convectively cooled
 - $-k(dT/dx)_{x_w} = h(T_\infty - T_s)$



- Those of you who remember your Diff. Eq. Will recognize the first BC as a Dirichlet type, the next two as Neumann type, and the last as being mixed!

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1-D, Steady State Heat Transfer

- In this chapter we will consider steady state heat transfer in one dimension.
- As a result we can set the left hand side of our governing equation to zero:

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + \dot{q} = 0$$

- The direction of heat flux will either be axial as in either the X or Z directions, or will be radial in the R direction.
- We will also consider situations which are really multidimensional, but which can be well modeled with 1-D approximations.

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The Planar Wall

- The first case of interest is the planar wall made of a single material as sketched.
- We will not consider heat generation at first, so $\dot{q} = 0$ and the governing equation is just:

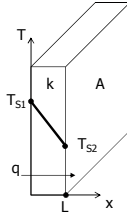
$$\nabla \cdot (k \nabla T) = \frac{d}{dx} \left(k \frac{dT}{dx} \right) = 0$$

- Another way to state this equation is:

$$q_x'' = -k \frac{dT}{dx} = \text{constant}$$

- Or

$$q_x = \frac{kA}{L} (T_{S1} - T_{S2})$$



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The Planar Wall (cont)

- A more complicated version of this problem is to combine multiple walls of different materials.
- For each wall section we have:

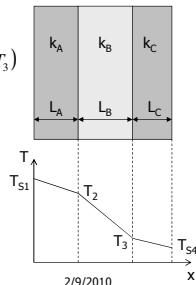
$$q_A = \frac{k_A A}{L_A} (T_{S1} - T_2) \quad q_B = \frac{k_B A}{L_B} (T_2 - T_3)$$

$$q_C = \frac{k_C A}{L_C} (T_3 - T_{S4})$$

- But, it must also be true that

$$q_A = q_B = q_C = q_x$$

since energy is not stored inside the wall.



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The Planar Wall (cont)

- This gives three equations, and 3 unknowns (q_x, T_2, T_3) if the two wall temperatures are given (T_{S1}, T_{S2}).
- Solving for the q_x yields:

$$q_x = \frac{(T_{S1} - T_{S4})}{\left(\frac{L_A}{k_A A} \right) + \left(\frac{L_B}{k_B A} \right) + \left(\frac{L_C}{k_C A} \right)}$$

- This result can be generalized to any multilayer composite wall by introducing the concept of **thermal resistance, R_t** .
- Thermal resistance plays the role in heat conduction as electrical resistance does in electrical conduction.

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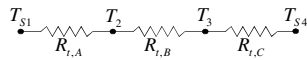
The Planar Wall (cont)

- Using an analogy to flow of electrical current, if the temperature change is the potential difference and q the flux rate, then:

$$R = \frac{\Delta V}{i} \Rightarrow R_t = \frac{\Delta T}{q_x} = \frac{L}{kA} \equiv \text{thermal resistance}$$

- The total resistance offered by this composite wall is similar to resistors in series:

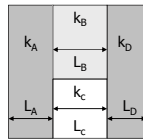
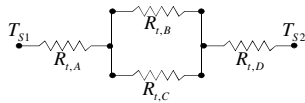
$$q_x = \frac{\Delta T}{R_{t,A} + R_{t,B} + R_{t,C}} = \frac{\Delta T}{\Sigma R_t} = \frac{\Delta T}{R_{tot}}$$



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The Planar Wall (cont)

- This electrical analogy can be extended to other situations such as the wall shown.
- This case resembles having resistors in series and parallel:



- The resistance in this case is give by:

$$R_{tot} = R_{t,A} + \frac{R_{t,B} R_{t,C}}{R_{t,B} + R_{t,C}} + R_{t,D}$$

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The Planar Wall (cont)

- And the net heat flux through the wall is just

$$q_x = \frac{\Delta T}{R_{tot}} \quad \text{or} \quad q_x'' = \frac{\Delta T}{AR_{tot}}$$

- A common way of expressing the effect of a composite wall is to use the **Overall Heat Transfer Coefficient, U**, defined by:

$$q_x \equiv UA\Delta T \quad \text{or} \quad q_x'' \equiv U\Delta T$$

- By comparison with the previous equation, it is obvious that :

$$U = \frac{1}{AR_{tot}}$$

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The Planar Wall (cont)

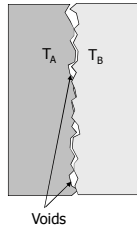
- Lastly, don't confuse the thermal resistance, R_t , with the "R" factor quoted for many common construction materials.
- The "R" factor used in the construction industry is defined by:

$$"R" \equiv \frac{L}{k}$$
 where L is the insulation thickness in inches and k the material conductivity in BTU inches/(hr ft² °F).
- Thus the "R" factor is the same as 1/U, but with British units.

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Contact Resistance

- One assumption that has been made in the previous development is that heat flow easily between the different layers of material.
- In fact, when two solids are in contact, their surface roughness results in voids separating the two materials.
- How easily heat flows between the two materials then depends upon the amount of direct contact between them and the fluid filling the gaps.



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Contact Resistance (cont)

- To account for this effect, the Contact Resistance is defined by:

$$q_x'' = h_c (T_A - T_B) \quad \text{or} \quad R_{t,c}'' = \frac{1}{h_c} = \frac{T_A - T_B}{q_x''}$$
- Unfortunately, it is very difficult to accurately predict the contact resistance and experimentation is usually necessary.
- Factors which effect the how much contact resistance there is are surface preparation, contact pressure and the void fluid.
- Two polished surfaces will make better contact, thus reducing resistance from the voids.

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Contact Resistance (cont)

- Joining two materials by force increases the contact area through the surface by the deformation of the faces under load - thus reducing resistance.
- Filling the voids with more conductive fluids such as oil or thermal greases reduces the resistance of the voids themselves.
- Finally, placing a thin foil of a highly ductile and conductive material such as lead or indium between two surfaces before joining them with force acts to effectively fill the voids.
- See the book for typical thermal contact resistance values.

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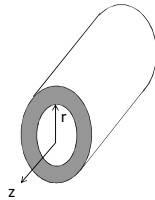
Radial Flow - Cylinders

- The next case of interest is the radial flow of heat in a cylindrical system. The best instance of this case is heat loss from a pipe.
- Without heat generation and with $k=\text{constant}$, the governing eqn. is:

$$\nabla \cdot (k \nabla T) = k \frac{d^2 T}{dr^2} + \frac{k}{r} \frac{dT}{dr} = 0$$

- Another way to state this equation is to multiply by r and combine terms:

$$rk \frac{d^2 T}{dr^2} + k \frac{dT}{dr} = \frac{d}{dr} \left(rk \frac{dT}{dr} \right) = 0$$



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Radial Flow - Cylinders (cont)

- This result is consistent with applying Fourier's Law to the case but letting the area be a function of r :

$$q_r = -kA \frac{dT}{dr} = -k2\pi rL \frac{dT}{dr} = \text{constant}$$

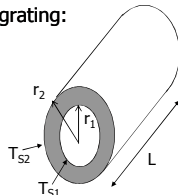
- By separating the variables and integrating:

$$\int_{T_{s1}}^{T_{s2}} \frac{dT}{dr} dr = \int_{r_1}^{r_2} \frac{-q_r}{2\pi rLk} dr$$

$$T_{s2} - T_{s1} = \frac{-q_r}{2\pi Lk} \ln \left(\frac{r_2}{r_1} \right)$$

- Or:

$$q_r = 2\pi Lk \frac{(T_{s1} - T_{s2})}{\ln(r_2 / r_1)}$$



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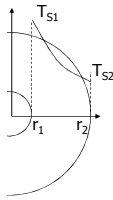
Radial Flow - Cylinders (cont)

- Or using the concept of thermal resistance:

$$q_r = \frac{(T_{s1} - T_{s2})}{R_t} \Rightarrow R_t = \frac{\ln(r_2 / r_1)}{2\pi Lk}$$

- Now think of the consequences of these results.

- First, the temperature does not vary linearly, rather it varies with the natural logarithm of r .
- Second, while the heat rate is constant, the heat flux is not, but varies as $1/r$.



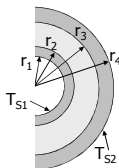
$$q_r = \text{constant} \quad q_r'' = \frac{q_r}{A} = \frac{\text{constant}}{2\pi rL}$$

Radial Flow - Cylinders (cont)

- For composite radial layers, the situation is analogous to the composite planar wall.
- Thus, for a pipe made of steel coated with a layer of fiberglass and then plaster:

$$q_r = \frac{(T_{s1} - T_{s2})}{R_{tot}}$$

$$R_t = \frac{\ln(r_2 / r_1)}{2\pi Lk_{steel}} + \frac{\ln(r_3 / r_2)}{2\pi Lk_{fiberglass}} + \frac{\ln(r_4 / r_3)}{2\pi Lk_{plaster}}$$



- Also, contact resistance will occur in radial systems as in planar walls.

Radial Flow - Spheres

- Radial heat flow in spheres is qualitatively similar to that in cylinders, but the flux area varies as r^2 .
- Thus, Fourier's Law is:

$$q_r = -kA \frac{dT}{dr} = -k4\pi r^2 \frac{dT}{dr} = \text{constant}$$

- And after integrating you get:

$$q_r = 4\pi k \frac{(T_{s1} - T_{s2})}{(1/r_1) - (1/r_2)} \quad R_t = \frac{(1/r_1) - (1/r_2)}{4\pi k}$$

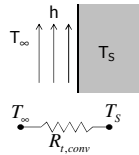
- Composite layers are handled as in the previous two cases.

Convection Boundaries

- All the cases considered thus far have assumed the surface temperatures are known.
- In general, however, the temperature of the surrounding fluid is known, and hopefully the convective heat coefficient, h .
- Using the concept of thermal resistance to Newton's Law of convective cooling shows that:

$$q_{x,conv} = hA(T_s - T_\infty) = \frac{(T_s - T_\infty)}{R_{t,conv}}$$

$$R_{t,conv} = \frac{1}{hA}$$



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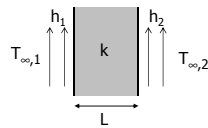
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Convection Boundaries (cont)

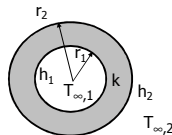
- Thus, for cooling on a planar wall:

$$q_s = \frac{(T_{\infty,1} - T_{\infty,2})}{\frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}}$$



- While for a pipe:

$$q_r = \frac{(T_{\infty,1} - T_{\infty,2})}{\frac{1}{h_1 2\pi r_1 L} + \frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{h_2 2\pi r_2 L}}$$



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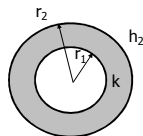
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Critical Radius of Insulation

- An unusual situation arises when applying insulation to a pipe of small radius.
- If only the insulation layer and the convective process on the outer surface are considered, the thermal resistance is:

$$R_{tot} = R_{t,cond} + R_{t,conv} = \frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{hL2\pi r_2}$$

- The conductive term above increases with r_2 , while the convective term decreases with r_2 .



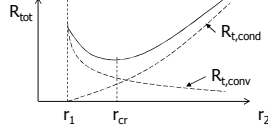
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Critical Radius of Insulation (cont)

- Thus, a plot of R_{tot} versus r_2 might look like:



- The radius at which R_{tot} reaches a minimum is called the critical radius of insulation, r_{cr} .
- Below this radius, the addition of insulation is counterproductive since the increase in the outer surface area (where convection occurs) is more significant than the insulating properties of the material.

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Critical Radius of Insulation (cont)

- To find the r_{cr} , we find the minimum of this curve:

$$\frac{dR_{tot}}{dr_2} = \frac{d}{dr_2} \left(\frac{\ln(r_2/r_1)}{2\pi Lk} + \frac{1}{hL2\pi r_2} \right) = \frac{1}{2\pi Lkr_2} - \frac{1}{hL2\pi r_2^2} = 0$$

- Or simply:

$$r_2 = r_{cr} = \frac{k}{h}$$

- Two notes:
 - First, for the insulation to be effective, the outer radius might have to be much greater than r_{cr} . As a result, thin pipes are often left un-insulated!
 - Second, for some cases $r_{cr} < r_1$, i.e. any insulation thickness is effective.

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Heat Generation - Plane Walls

- Let's revisit the plane wall case, but this time include heat generation, \dot{q} .
- The governing equation in this case is:

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

- Integrating this equation twice yields a quadratic equation in x with two unknown constants of integration:

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1x + C_2$$

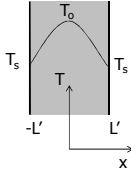
- The constants must be determined by applying the appropriate boundary conditions.

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Heat Generation - Plane Walls (cont)

- Let's consider the simplest case possible, that where the temperature on both walls are equal at T_s .
- Symmetry dictates that $C_1=0$ and to have $T=T_s$ at $x=L'$ (half thickness):

$$C_2 = \frac{\dot{q}}{2k} L'^2 + T_s = T_0$$
- Note that C_2 is also equal to the temperature at the center of the wall, T_0 .
- Thus the temperature variation in the wall is given by:



$$T(x) = \frac{\dot{q}}{2k} (L'^2 - x^2) + T_s \quad \text{or} \quad \frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L'} \right)^2$$

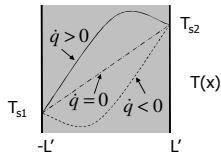
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Heat Generation - Plane Walls (cont)

- In the more general case where the two surface temperatures are at different values, $T_{s,1}$ and $T_{s,2}$, the solution is:

$$T(x) = \frac{\dot{q}L'^2}{2k} \left(1 - \frac{x^2}{L'^2} \right) + \frac{T_{s,2} - T_{s,1}}{2} \frac{x}{L'} + \frac{T_{s,1} + T_{s,2}}{2}$$

- Which looks like:



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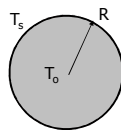
Heat Generation - Radial Systems

- Radial systems are very similar. The governing equation is:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{\dot{q}}{k} = 0$$

- Which after integrating twice yields:

$$T(r) = -\frac{\dot{q}}{4k} r^2 + C_1 r + C_2$$



- Once again, symmetry requires $C_1 = 0$ and C_2 must equal centerline temperature and:

$$C_2 = \frac{\dot{q}}{4k} R^2 + T_s = T_0$$

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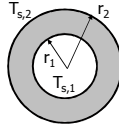
Heat Generation - Radial Systems (cont)

- Substitution then yields the result that:

$$T(r) = \frac{\dot{q}R^2}{4k} \left(1 - \frac{r^2}{R^2} \right) + T_s \quad \text{or} \quad \frac{T(r) - T_s}{T_0 - T_s} = \left(1 - \frac{r^2}{R^2} \right)$$

- In the more general case shown below, the solution is:

$$T(r) = \frac{\dot{q}}{4k} \left[(r_2^2 - r^2) + (r_2^2 - r_1^2) \frac{\ln(r/r_2)}{\ln(r_1/r_2)} \right] + (T_1 - T_2) \frac{\ln(r/r_2)}{\ln(r_1/r_2)} + T_2$$



- Care to prove it?

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Heat Generation with Convection

- Now let's add surface convection into these two problems.
- First, we would like to find the total heat flux out of the surface. From our previous results:

<u>Plane Wall</u>	<u>Radial System</u>
$q_s = -kA \frac{dT}{dx} \Big _{x=L}$	$q_s = -kA \frac{dT}{dr} \Big _{r=R}$
$q_s = -kA \left(\frac{-\dot{q}L'}{k} \right)$	$q_s = -k2\pi RL \left(\frac{-\dot{q}R}{2k} \right)$
$q_s = \dot{q}AL' = \dot{q}V'$	$q_s = \dot{q}\pi R^2L = \dot{q}V'$

(V' = 1/2 total volume) AE 301 Aerodynamics I 101 2/9/2010

Heat Generation with Convection (cont)

- In retrospect, this result makes sense - the heat flux out must equal the heat generated within!
- However, this heat to the wall must be carried away by convection (or radiation, possibly).
- Thus:

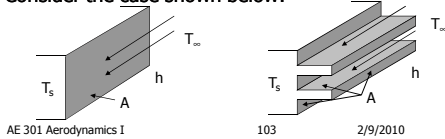
<u>Plane Wall</u>	<u>Radial System</u>
$\dot{q}AL = hA(T_s - T_\infty)$	$\dot{q}\pi R^2L = h2\pi RL(T_s - T_\infty)$
$T_s = \frac{\dot{q}L'}{h} + T_\infty$	$T_s = \frac{\dot{q}R}{2h} + T_\infty$

- Notice how these results are independent of the internal temperature profile!

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Extended Surfaces

- From considering the critical radius of insulation, we saw the importance of surface area when determining the total heat transfer in a combined conduction/convection system.
- In many surfaces the rate of heat transfer can be enhanced by increasing surface area through the use of extended surfaces or "fins"
- Consider the case shown below:

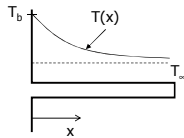


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Extended Surfaces (cont)

- The addition of fins can greatly increase the total convective surface area.
- However, realize that this added surface area is not as effective as the original since T decreases from the base temp, T_b , along the length of the fin!
- Since the heat loss from the fin is proportional to the difference $(T(x) - T_\infty)$, the base of the fin is more effective than the root.
- To calculate the net heat flux, we need to find T as a function of x.



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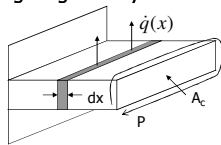
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Extended Surfaces (cont)

- This problem is at least 2 dimensional. However a very good approximation involves considering heat flux in only one dimension - base to tip.
- Heat flux normal to this direction can be modeled as heat loss (negative \dot{q}) which is due to convection.
- To see this, consider the following fin geometry:
- Let A_c be the fin tip area and P the length of the perimeter of this area.
- Then $\dot{q}dV = -hdA(T(x) - T_\infty)$

or $\dot{q}A_c dx = -hPdx(T(x) - T_\infty)$



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Extended Surfaces (cont)

- The governing equation for this quasi 1-dimensional problem is then:

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = \frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T(x) - T_\infty) = 0$$

- This problem can be notationally simplified by letting:

$$\theta(x) = (T(x) - T_\infty) \quad \text{and} \quad m^2 = \frac{hP}{kA_c}$$

- To get: $\frac{d^2\theta}{dx^2} - m^2\theta(x) = 0$

which has the general solution:

$$\theta(x) = Ae^{mx} + Be^{-mx}$$

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Extended Surfaces (cont)

- To solve for the two constants A and B, we need two boundary conditions.

- At the base, $T_{x=0} = T_0$ or $\theta_{x=0} = \theta_0 = (T_0 - T_\infty)$

- At the other end, the tip, the best B.C. would include the effect of tip heat convection, or:

$$q_{conduction}|_{x=L} = q_{convection}|_{x=L} \Rightarrow -kA_c \left. \frac{dT}{dx} \right|_{x=L} = hA_c(T_{x=L} - T_\infty)$$

- In practice, it is more reasonable that no heat is lost through the tip both because A_c is small and the temperature there is close to T_∞ . I.e.

$$\left. \frac{dT}{dx} \right|_{x=L} = \left. \frac{d\theta}{dx} \right|_{x=L} = 0$$

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Extended Surfaces (cont)

- With this "insulated end" B.C., the solution for A and B yields:

$$\frac{\theta}{\theta_0} = \frac{e^{mx}}{1 + e^{2mL}} + \frac{e^{-mx}}{1 + e^{-2mL}} = \frac{\cosh[m(L-x)]}{\cosh[mL]}$$

- Of course, the temperature distribution is secondary. What we really want is the fin heat loss rate:

$$q_f = \int_0^L hP(T - T_\infty) dx = \int_0^L hP\theta dx = \frac{hP\theta_0}{m} \tanh(mL)$$

$$\text{or } q_f = \sqrt{hPkA_c}(T_0 - T_\infty) \tanh\left(L\sqrt{\frac{hP}{kA_c}}\right)$$

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Fin Effectiveness and Efficiency

- When designing fins, or deciding what fin to use, it is useful to have some parameters for comparison.
- For fins, there are two parameters commonly used: effectiveness and efficiency.
 - Fin effectiveness, ϵ_f : the ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin.
 - Fin efficiency, η_f : the ratio of the fin heat transfer rate to the ideal case where $T(x) = T_0$ all along the fin length.
- Fin effectiveness will tell us how much more effective having the fin is over not having it.
- Fin efficiency will tell us whether it makes the optimum use of material.

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Fin Effectiveness and Efficiency (cont)

- For the case just considered, the effectiveness would be:

$$\epsilon_f = \frac{q_f}{hA_c(T_0 - T_\infty)} = \sqrt{\frac{kP}{hA_c}} \tanh(mL)$$

- Note that for long fins ($\tanh(mL) \rightarrow 1$), then three things improve fin effectiveness:
 - A high material conductivity. Pretty obvious!
 - A low fluid conductivity. This does not mean that you want h low, just that fins are more commonly used when h is small as in natural convection and/or gasses versus liquids.
 - A large ratio of tip area to tip perimeter, A_t/P . Thus a wide, thin fin is preferable. Another way to say this is that you would rather have two thin fins than one twice as thick

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Fin Effectiveness and Efficiency (cont)

- The fin efficiency for this case is:

$$\eta_f = \frac{q_f}{hPL(T_0 - T_\infty)} = \frac{\tanh(mL)}{mL}$$

- This result indicates two things.
- The highest efficiency, 1, is achieved for a short fin since $\tanh(mL) \rightarrow mL$ as L is decreased.
- The fin efficiency decreases to a value of zero as the length increases.
- This is because the region near the tip, where $T \rightarrow T_\infty$, does not contribute as much as the base sections.

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Other Fin Shapes

- The book gives the results for span efficiency for other type fins on pages 45 and 46.
- Note that the first case shown is the one we just solved, but using a corrected length, L_c , not L .
- This slight change improves the comparison to experiment since the tip is not really insulated!
- Also note that all these solutions just provide the fin efficiency and the fin wetted area, A_f .
- To get fin effectiveness, use:

$$\epsilon_f = \frac{A_f}{A_b} \eta_f \quad A_b \equiv \text{fin base area}$$

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Fin Thermal Resistance

- When dealing with complex thermal flow situations, it is useful to fall back to the electrical analogy.
- For fins, the thermal efficiency is defined by:

$$R_f = \frac{(T_0 - T_\infty)}{q_f}$$

- However, also note that the fin effectiveness is related to the fin thermal efficiency by:

$$\epsilon_f = \frac{R_b}{R_f} \quad R_b \equiv \frac{1}{hA_b}$$

- Thus use the chart to get efficiency, then from that find the fin effectiveness and the R_f .

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Overall Surface Effectiveness

- The study of fins so far has concentrated on a single fin. Now we want to extend this to multiple fins and the base surface area between them.

- If we define the overall surface efficiency by:

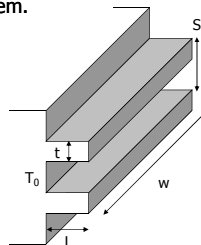
$$\eta_o = \frac{q_i}{q_{\max}} = \frac{q_i}{hA_b\theta_0}$$

$q_i \equiv$ total heat transfer $A_b \equiv$ total area

- With just a little bit of work, it can then be shown that:

$$\eta_o = 1 - \frac{NA_f}{A_b} (1 - \eta_f)$$

$N \equiv$ # of fins



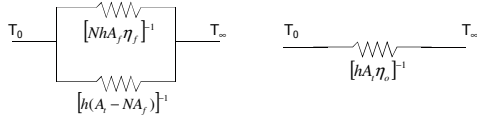
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Overall Surface Effectiveness (cont)

- Finally, we can find the overall thermal resistance by:

$$R_{t,o} = \frac{\theta_o}{q_t} = \frac{1}{hA_s \eta_o}$$

- With the equivalent thermal networks:



- One thing to keep in mind - we have assumed h is a constant. But if you space the fins too closely, h will actually decrease due to blockage effects!

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