Two-Dimensional Heat Transfer

- Thus far, we have only considered cases where heat fluxes (or temp. variations) are in one main direction.
- Lets go beyond this and consider cases of 2-D heat transfer in both simple and complex geometries.
- We won't do 3-D, but it uses the same methods as 2-D - just harder.
- The governing equation for multidimensional, steady heat conduction is: $\nabla^2 T = -\frac{\dot{q}}{k}$
- Before we begin to consider how to solve this, let's first just consider the properties of the Laplacian operator.

The Laplacian Operator

- In 2-D Cartesian Coordinates, the Laplacian operator is nothing more than: $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{\dot{q}}{k}$
- This operator is a great "smoothing" function
  - The solution of this equation is smooth and continuous, except possibly at discontinuous boundaries.
  - The maxima and minima of the solution occur on the boundaries (if $\dot{q}$ = 0)
  - This equation represents many common physical problems such as electromagenetic flux, gravitational potential, inviscid and incompressible flow, and the deflection of elastic membranes.
  - As a result, this equation has been studied for many years.

The Laplacian Operator (cont)

- The book considers 3 ways of solving this equation.
- Analytic mathematical solutions:
  - Fairly easy to obtain for simple geometries and simple BC's.
  - A horror to find on complex geometries and/or BC's.
  - Gives complete, continuous solutions of T(x,y)
- Graphical solutions:
  - Approximate method which exploits "smoothness" feature.
  - Relatively painless - but more of an art than a science.
- Numerical solutions:
  - Hard to program, but once the code exists, pretty easy to use for many different cases.
  - Only provides solutions of T and discrete points.

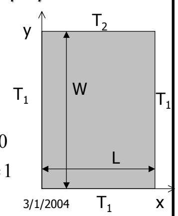
Analytic 2-D Solutions

- Consider the simply geometrical case as shown - a retangular region the same temperature on 3 sides, but different on the fourth.
- To solve this problem, we use the technique of Separation of Variables. First, let's simplify the BC's by defining

$$\theta(x,y) = \frac{T(x,y) - T_1}{T_2 - T_1}$$

- With this change, our governing eqn. and BC's for no heat generation are:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \qquad \theta(0,y)=0 \quad \theta(x,0)=0 \qquad \theta(L,y)=0 \quad \theta(x,W)=1$$



Analytic 2-D Solutions (cont)

- Now, assume that the solution will have the form:

$$\theta(x,y) = X(x)Y(y)$$

- So that, after substituting into the governing eqn:

$$Y(y)\frac{d^2X(x)}{dx^2} + X(x)\frac{d^2Y(y)}{dy^2} = 0$$

- Or

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2$$

- Where λ must be a constant since the first term is only a function of x and the second is only a function of y!

Analytic 2-D Solutions (cont)

- Thus, the single 2nd order PDE becomes two 1st order ODE's:

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \qquad \frac{d^2Y}{dy^2} - \lambda^2 Y = 0$$

- Which have the general solutions for X(x) and Y(y):

$$X(x) = C_1\cos(\lambda x) + C_2\sin(\lambda x) \qquad Y(y) = C_3 e^{-\lambda y} + C_4 e^{\lambda y}$$

- Or, the general solution with BC's is:

$$\theta(x,y) = \left(C_1\cos(\lambda x) + C_2\sin(\lambda x)\right)\left(C_3 e^{-\lambda y} + C_4 e^{\lambda y}\right)$$

$$\theta(0,y)=0 \qquad \theta(x,0)=0$$
$$\theta(L,y)=0 \qquad \theta(x,W)=1$$

Analytic 2-D Solutions (cont)

- To satisfy the first BC, $\theta(0,y)=0$, it must be true that $C_1 = 0$.
- To satisfy the second, $\theta(x,0)=0$, we must have $C_4 = -C_3$
- To satisfy the second, $\theta(L,y)=0$, the product λL must be a multiple of π. Or

$$\lambda = \frac{n\pi}{L} \qquad n = 1,2,3,4.....$$

- Thus, we don't have a single solution, but a whole series of them which can be superimposed:

$$\theta(x,y) = \sum_{n=1}^{\infty} C^n{}_2 C^n{}_3 \sin\left(\frac{n\pi x}{L}\right)\left(e^{-n\pi y/L} - e^{n\pi y/L}\right)$$

Analytic 2-D Solutions (cont)

- To simplify this a little, lets introduce the sinh function and group all the constants into a single term:

$$C^n{}_2 C^n{}_3\left(e^{-n\pi y/L} - e^{n\pi y/L}\right) = C^n \sinh\left(n\pi\, y/L\right)$$

- To get:

$$\theta(x,y) = \sum_{n=1}^{\infty} C^n \sin\left(\frac{n\pi\, x}{L}\right)\sinh\left(\frac{n\pi\, y}{L}\right)$$

- To get this last constant (or series of constants!) we apply the final BC: $\theta(x,W)=1$

$$\theta(x,W) = 1 = \sum_{n=1}^{\infty} C^n \sin\left(\frac{n\pi\, x}{L}\right)\sinh\left(\frac{n\pi\, W}{L}\right)$$

Analytic 2-D Solutions (cont)

- This last equation appears impossible to solve. However, there is a little mathematical trick.
- Let's take this equation, multiply both sides by sin(mπx/L) and integrate from 0 to L.

$$\int_0^L \sin\left(\frac{m\pi\, x}{L}\right)dx = \sum_{n=1}^{\infty} C^n \sinh\left(\frac{n\pi\, W}{L}\right)\int_0^L \sin\left(\frac{n\pi\, x}{L}\right)\sin\left(\frac{m\pi\, x}{L}\right)dx$$

- The trick is that the integral on the RHS is zero UNLESS m=n, when it becomes L/2. This result is related to the concept of <u>orthogonal functions</u>. Thus:

$$\int_0^L \sin\left(\frac{n\pi\, x}{L}\right)dx = C^n \frac{L}{2}\sinh\left(\frac{n\pi\, W}{L}\right)$$

Analytic 2-D Solutions (cont)

- Finally, the integral on the LHS will be zero for even values of n. For odd values it has the solution 2L/nπ.

$$\frac{2L}{n\pi} = C^n \frac{L}{2} \sinh\left(\frac{n\pi W}{L}\right) \qquad n = 1, 3, 5, 7....$$

- Or, playing algebra games:

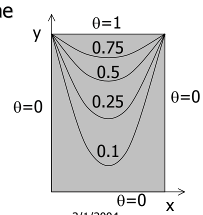
$$C^n = \left[(-1)^{n+1} + 1\right] \frac{2}{n\pi \sinh\left(\frac{n\pi W}{L}\right)} \qquad n = 1, 2, 3, 4....$$

- And our final solution for this problem is:

$$\theta(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\left[(-1)^{n+1} + 1\right]}{n} \sin\left(\frac{n\pi x}{L}\right) \frac{\sinh(n\pi y / L)}{\sinh(n\pi W / L)}$$

Analytic 2-D Solutions (cont)

- All this effort. And this was a simple problem!
- Also, while this solution has the value of being continuous in x and y, it also involves an infinite sum.
- In practice, we might only include the first 10 or so terms, but where we truncate the sum can effect the solution.
- By the way, if we evaluate this sum, the solution for θ(x,y) looks something like:

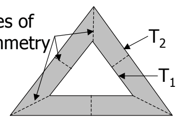


Graphical Method for 2-D Conduction

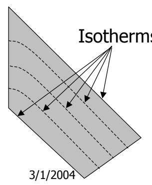
- The second method of 2-D conduction analysis is called the Graphical Method.
- In this method, isotherms (lines of constant T) and adiabats (line of constant heat flux) are sketched based on simple geometric rules and a desire to achieve “smoothness”.
- As a result, this method is approximate - but it can be fairly accurate if the “artist” has experience.
- The method is also relatively quick - an advantage if only preliminary results are needed and the boundaries have known temperatures.

Graphical Method for 2-D Conduction

- Begin any problem by identifying lines of symmetry - which should also be adiabats!



- Begin any problem by identifying lines of symmetry - which should also be adiabats!
- Next sketch in “smooth” isothermal lines, making sure they are perpendicular to the adiabats.

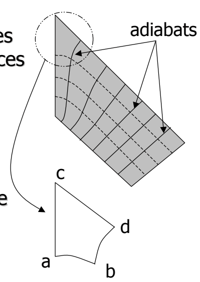


Graphical Method (cont)

- Draw adiabats in as lines perpendicular at every crossing with the isotherms.
- Finally begin iterating on the lines trying to obtain equal arc distances on opposing sides using:

$$\Delta x \equiv \frac{ab+cd}{2} \approx \Delta y \equiv \frac{ac+bd}{2}$$

- When finished, you have a graphical (qualitative) idea of the heat flux patterns.



Graphical Analysis (cont)

- To get quantitative results from the graphical diagram, use the following procedure.
- If properly drawn, there will be M lanes (space between adiabats) of equal heat heating rate.
- Similarly, there will be N temperature steps (space between isotherms) of equal temperature increments.
- If the plot was drawn correctly, with balanced side lengths, then the total heat flux is just:

$$\frac{q}{l} \approx \frac{M}{N} k \Delta T_{1-2}$$

  - where l is the width in the 3rd dimension.

The Conduction Shape Factor

- A simply way to represent analytical or graphical results for 2-D (or 3-D) heat conduction is in the form of a shape factor, S: $\boxed{q = Sk\Delta T_{1-2}}$
- This representation depends upon having two well defined surface temperatures, T_1 and T_2.
- Also not the the thermal resistance for this case is just: $R_{t,cond} = \frac{1}{Sk}$
- Some common shape factors are giving in Table 3-1 of the text.
- The process is simple: find the desired geometry, calculate S, find q or R from the above eqns.

Numerical Solutions

- The governing equation for multidimensional heat conduction may also be solved using a number of different numerical solutions.

$$\nabla^2 T = -\frac{\dot{q}}{k}$$

- The book discusses the classical Finite Difference solution method
- However, the FEHT program provided uses the Finite Element method of analysis.
- We will not worry about how to set up the solutions, just how to apply the program to different cases and conditions.