Linear Elasticity

Objectives

In this lab you will measure the Young's Modulus of a steel wire. In the process, you will gain an understanding of the concepts of stress and strain.

Equipment

- Young's Modulus apparatus
- Weights
- Micrometer
- Laboratory balance

Introduction

When a solid is stretched, compressed, or deformed in some other way, it is said to contain a *strain*. To produce a strain in a solid one must apply some force to the solid, for example by pressing on its surface. Note that the word strain refers to the deformation, not to the force required to produce that deformation. The strain thus produced, will in general vary as a function of position within the body of the object. Thus, at every point within the body of the object, we can quantify the strain both by the magnitude of the stretching of the body at that point and by the direction and way in which it is stretching. At some points within the body the strain may be a pure elongation or compression of the material while at other points the strain may be a shearing deformation. To describe the different kinds of stretching fully, we need to define the strain as a 3x3 matrix as follows. Consider a point $\vec{r}$ within the body of the object. At any such point, the portion of the solid at that point will be displaced from its equilibrium position by the applied forces. Let $\vec{u}(\vec{r})$ be the displacement of the solid at point $\vec{r}$ from its equilibrium position. The strain, $\varepsilon(\vec{r})$ is then given by the following matrix, known as the strain tensor.

$$\varepsilon(\vec{r}) = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} \tag{1}$$

where all derivatives are evaluated at $\vec{r}$. First look at the diagonal elements, for example the ε_{11} (top-left) component

$$\varepsilon_{11} = \frac{\partial u_x}{\partial x} = \lim_{\Delta x \to 0} \left(\frac{\Delta u_x}{\Delta x} \right) . \tag{2}$$

As we move in the *x*-direction by a distance Δx, the displacement of the material from equilibrium *changes* by an amount $\Delta\vec{u}$. In other words, the portion of material in the region Δx must be *stretched*. Δu_x is simply the *x*-component of the displacement change over the portion of material of length Δx. So the *relative* length change of the material in the region Δx is $\frac{\Delta u_x}{\Delta x}$. Thus $\varepsilon_{11}(\vec{r})$ tells us the relative length change of the material in the *x*-direction in an infinitesimal region around the point $\vec{r}$. Similarly, the other diagonal components tell us about the relative length change (or "stretch") in the *y* and *z*-directions.

What about the off-diagonal terms? They tell us about shear. Shear strain is what you get when you move the surface of an object parallel to the surface itself. For example, the component ε_{12} tells us how much the *x*-component of displacement changes as we move in the *y*-direction, i.e. how much *xy*-shear there is at that point. In this lab, we won't be concerned with shear.

Taken together, the 3x3 matrix of strain components are known as the strain tensor. They are a field (in other words, they depend on position) but not the usual kind of field with which we are hitherto familiar with. All fields we've seen so far are either scalar fields (e.g. temperature as a function of position) or vector fields (e.g. velocity of a fluid as a function of position). This is a new kind of field, a *tensor field*. Tensor fields feature prominently in Electromagnetism, Classical and Quantum field theory and General Relativity. For this reason, it is a good idea to become familiar with the strain tensor field.

Now, the stretching and shearing described by the strain tensor, inevitably leads to intra-molecular forces that resist this strain. These intra-molecular forces are of course highly non-linear in general but for *small* deformations, the intra-molecular force is simply proportional to the magnitude of the strain. Averaged over some tiny region in the material (that nonetheless contains huge numbers of molecules) we call these forces "linear elastic forces" and we call materials exhibiting linear elastic forces, linear elastic materials. The elastic force per unit area (across which it acts) is known as *stress*. Just as strain is a tensor, so is stress. In a linear elastic material, the components of stress, σ_{ij} are proportional to the components of strain ε_{ij}. Obviously, there are a lot of proportionality factors, each of which could in principle be different. However, in homogenous, isotropic materials the number of proportionality factors is greatly reduced. In fact, there are only two independent components of proportionality, called *Young's Modulus* and *Poisson's Ratio* and they are constant material properties (like density for example or specific heat capacity).

The Young's modulus is given by the ratio of stress to strain for the stretching-type strains only (i.e. strains lying along the diagonal of the strain matrix). In other words, there is no shear involved. For example, the Young's Modulus of a particular metal could be measured by subjecting a wire made of this metal to a stretching force, as we do in this lab. The ratio of the stress being applied to the resulting strain in the wire is the Young's Modulus, *Y*. (By the way, Poisson's Ratio is the ratio of the relative length change due to stretching to the relative narrowing of the wire due to the stretching. We won't be concerned with Poisson's ratio in this lab.)

In the case of a stretched wire, the stress in the wire will be uniform and equal to the ratio of the applied force to the cross-sectional area of the wire

$$\sigma = \frac{F}{A} \tag{3}$$

The resulting deformation will be almost exclusively be a stretching of the wire along the length of the wire (no stretching along a diameter). So the only significant strain component will be the one corresponding to displacements along the direction of the wire (call it the *z*-direction). So we can expect

$$\varepsilon(\vec{r}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix} \tag{4}$$

Since the amount by which the wire stretches is the same everywhere along the wire (assuming a uniform wire diameter), the strain is just a constant $\varepsilon_{33} = \frac{\partial u_z}{\partial z} = \frac{\Delta u_z}{\Delta z}$ where Δu_z is the change in the displacement of the wire between the top and the bottom of the wire. In other words (since the top of the wire, at the attachment point will have no displacement) Δu_z is simply the total extension ΔL of the wire. The distance Δz is just the total length of the wire, L.

So, Young's modulus will be given by

$$Y = \frac{\sigma}{\varepsilon} = \frac{F/A}{\Delta L/L}. \tag{5}$$

<u>Some Notes</u>:

- ΔL (and therefore also the strain) will be negative for compression stresses and positive for tension stress.

- Recall that Hooke's law describes the amount of change in the length of a spring as being directly proportional to the force applied to it:

$$F = -k\,\Delta L \tag{6}$$

Where F is the force exerted, ΔL is the amount of elongation or compression and k is a constant of proportionality that describes the stiffness of the spring. (The minus sign is a historical convention applied in to allow k to be positive.)

- When considering the length change of a material, the Young's Modulus relates the stress and the strain in the same way that Hooke's constant relates elongation to force

$$\sigma = Y\varepsilon \tag{7}$$

In the case of a wire, Hooke's constant of the wire is related to the Young's Modulus as follows

$$k = -\frac{F}{\Delta L} = -\frac{A\sigma}{L\varepsilon} = -\frac{A}{L}\left(\frac{\sigma}{\varepsilon}\right) = -\frac{A}{L}Y \tag{8}$$

Procedure

We will measure the change in length of a wire as it supports an ever increasing load. In the Young's modulus apparatus illustrated in Figure 1 below, a wire is suspended from the top of the frame. Masses are added to the mass-hanger attached to the bottom of the wire in order to stretch the wire. To ensure a well-functioning apparatus, you will need to familiarize yourself with the bubble-level and micrometer assembly. Figure out how it works! In particular, figure out how the various parts move relative to one another and why.

1. Make a labeled diagram (not "just" an artist's sketch) of the bubble-level and micrometer assembly. In a few sentences explain how the mechanism works, referring to the various parts of the labeled diagram. Make sure you note which parts are free to move, pivot, turn etc.

2. Once you are sure you understand how the mechanism works, check that it is properly oiled and that the screws, particularly the set screws supporting the pivoting plate are correctly adjusted. Make sure the cylinder holding the wire collet moves freely within the hole through the base plate (below the pivoting plate) and that it is not catching on the pin that keeps the cylinder from turning.

3. Have a careful look at your wire. Make notes on any kinks, twists or other imperfections you may see.

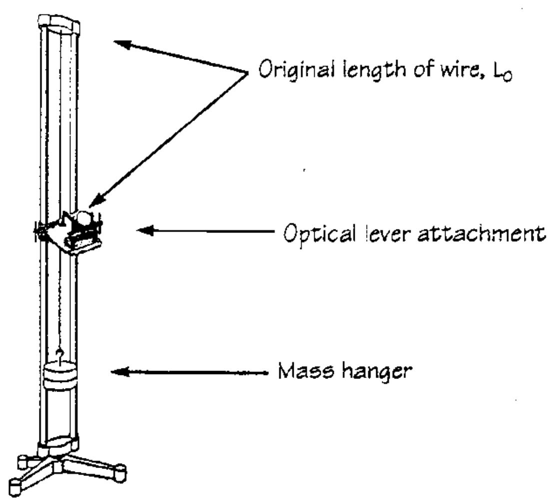

Figure 1: Young's Modulus Apparatus. The wire to be tested is fastened at the top of the apparatus and a mass-hanger is fastened to the wire at the bottom. Masses are added and removed from the mass hanger to vary the force that produces the stress. In our setup, a micrometer and bubble level assembly replaces the optical lever attachment indicated here.

4. Measure the wire diameter with a micrometer, avoiding any kinks in the wire. Make several measurements at various positions along the wire and find the average. Note that wires can be slightly become slightly elliptical in the drawing process during manufacture. Make any additional measurements you may require in order to take this possibility into account. Then calculate the wire area. *Penultimate Reminder*: Make sure you are calculating uncertainties in

measured and derived quantities! You will need to do the error propagation correctly following the methods described in the section “Error in Measurements – Theory.”

5. With one kilogram of weight on the wire (e.g. just the large mass hanger itself), measure the total length L of the wire.

6. With 1 kg still suspended by the wire, make a few practice measurements of the micrometer reading. Then, if needed adjust the vertical location of the sled holding the micrometer and bubble assembly until it is approximately level with 6 kilograms of suspended mass. Make a few more practice measurements and then remove all the suspended mass including the mass hanger.

7. First we will do a rough measurement run. Starting with a suspended mass of 1 kg (just the large mass hanger), measure and record the wire extension as a function of suspended mass in increments of 2 kg, until the suspended mass reaches 11 kg. In any case, do not exceed a total suspended mass of 12 kg as the wire may break. Note that the wire extension at a suspended mass of M is simply the micrometer reading at that suspended mass minus the initial micrometer reading (in this case with 1 kg suspended).

8. By hand, make a plot of wire extension ΔL versus suspended mass M. If it is approximately linear, draw a by-eye best-fit line through the data. Also, draw “maximum reasonable slope” and “minimum reasonable slope” lines through the data. Using Eqn. 5 above as a starting point, calculate the Young’s modulus of the wire from the slope of the best-fit line. Using the maximum and minimum reasonable slope lines, calculate an approximate uncertainty in your value for the Young’s Modulus. You may find it interesting at this point to compare your value to the handbook value of Young’s Modulus for the wire material that you are using.

9. Now we will do a much more careful measurement run. Starting with a mass of approximately 0.2 kg , measure the wire extension as a function of suspended mass between 0.2 kg and approximately 1.0 kg in increments of no more than 0.05 kg (50 g). When you reach one kilogram, change the size of the increments to 0.2 kg, until you reach a total mass of 2 kg. Then change the size of the increments to 0.5 kg until you reach 12 kg. Do not exceed a total suspended mass of 12 kg as the wire may break. Record the mass and corresponding length

Suspended Mass, M (kg)	Length Change ΔL (mm)	Stress σ (N/m^2)	Strain ε

changes to a table with column headings similar to the one below.

Final Reminder: Remember to estimate and propagate uncertainties to *each* entry!

10. On millimeter-ruled graph paper, plot stress versus strain with strain along the *x*-axis and stress along the *y*-axis.

Questions

1. Use Eqn. 5 to find the S.I. units of Y. (Show your work!)

2. In parts 9 and 10 above, it's likely that data for very low suspended mass is qualitatively different from that for the higher suspended mass. Come up with hypothesis for what is going on and come up with a way to test this hypothesis. If time permits carry out that test.

3. Using the data from part 9, draw a best-line fit onto your data for masses above about 2 kg (i.e. in the linear region of your data). Find a value for the Young's Modulus from the slope. Find the uncertainty in the Young's Modulus by drawing "maximum reasonable slope" and "minimum reasonable slope" lines (as in part 8 above). Since you have more points in this data set than in the data from part 7, you should have a *smaller* range of slopes that you feel are reasonable guesses at the true slope. Therefore, you should have *lower* uncertainty in the Young's modulus. (This assumes that the spread of data points is dominated by uncertainties.)