AE301 Aerodynamics I

UNIT B: Theory of Aerodynamics

ROAD MAP . . .

- **B-1**: Mathematics for Aerodynamics
- **B-2**: Flow Field Representations
- **B-3**: Potential Flow Analysis
- **B-4**: Applications of Potential Flow Analysis

Unit B-2: List of Subjects

- Concept of Streamline
- Strain and Angular Velocity
- Angular Velocity and Vorticity
- Rotational v.s. Irrotational Flows
- Circulation of a Flow Field
- Kutta-Joukowski Theorem
- Stream Function
- Velocity Potential
- Inviscid and Incompressible Flow Equations
- Laplace’s Equation
CONCEPT OF STREAMLINE

A streamline is a line whose tangent at any point is in the direction of the velocity vector at that point. Mathematically, this means: \( \mathbf{dS} \times \mathbf{V} = \mathbf{0} \).

Let us examine this equation in 3-D Cartesian coordinate system:
\[
\mathbf{dS} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \quad \text{and} \quad \mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}
\]

Therefore,
\[
\mathbf{dS} \times \mathbf{V} = dx \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dy & dz & \ \end{vmatrix} + dy \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dz & dx & \ \end{vmatrix} + dz \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u & v & w \end{vmatrix}
\]

\[
= \mathbf{i} (w dy - v dz) + \mathbf{j} (u dz - w dx) + \mathbf{k} (v dx - u dy)
\]

EQUATION OF STREAMLINE
Strain and Angular Velocity

THE MOTION OF A FLUID ELEMENT

The motion of a fluid element along a streamline is a combination of “translation” and “rotation.” In addition, the shape of the element can become “distorted” (“angular deformation”) as it moves.

- The presence of “viscosity” causes both “rotation” and “angular deformation,” and the flow field is defined as “rotational flow.”
- If the viscosity within the flow field is ignored (“inviscid flow”), the motion of a fluid element can be simplified only for “translation”: this is called, “irrotational flow.”

ANGULAR VELOCITY OF A FLUID ELEMENT (IN 2-D CARTESIAN: X-Y PLANE)
Angular Velocity and Vorticity

Angular Velocity Vector of a Fluid Element (3-D Cartesian)

Repeating a similar procedure in y-z and x-z planes, one can obtain:

\[ \omega_x = \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \]  (y-z plane)  
\[ \omega_y = \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \]  (x-z plane)

The angular velocity is a vector (thus, 3-components in 3-D Cartesian coordinate system), such that:

\[ \hat{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} = \frac{1}{2} \left[ \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k} \right] \]  (angular velocity vector)

- Let’s recall: the “curl” of the velocity field is . . . \( \text{curl} \left( \mathbf{V} \right) = ? \)

Vorticity

- In aerodynamics, the curl of the velocity field is equal to the twice of the angular velocity.

- Mathematically: \( \xi = 2\hat{\omega} = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k} = \text{curl} \left( \mathbf{V} \right) = \nabla \times \mathbf{V} \)

- This is defined as: “vorticity” of a given flow field, in aerodynamics.

\[ \xi = \text{curl} \left( \mathbf{V} \right) = \nabla \times \mathbf{V} \]  (vorticity of flow field)
ROTATIONAL V.S. IRROTATIONAL FLOW

- We can categorize “flow field” of aerodynamics into two distinctively different types, based on the vorticity in the flow field:

1. If $\nabla \times \mathbf{V} \neq 0$ at every point in a flow, the flow is called rotational. This implies that the fluid elements have a finite angular velocity.

2. If $\nabla \times \mathbf{V} = 0$ at every point in a flow, the flow is called irrotational. This implies that the fluid elements have no angular velocity; rather, their motion through space is a pure translation.

“IDEAL” (OR “POTENTIAL”) FLOW FIELD

- If the flow field is “irrotational,” the governing equation of the flow field can be significantly simplified. As a result, one can model and analyze basic behaviors of the flow field by simple mathematical formulations.

- If the flow field is:
  (i) steady,
  (ii) no body force,
  (iii) incompressible,
  (iv) inviscid,
  (v) and irrotational (because “inviscid”):
  the flow field is called, “ideal” or “potential” flow field.

- Ideal (or potential) flow fields can be analyzed mathematically (100% pure theory). This is commonly referred to be: “theoretical aerodynamics” by “potential flow field analysis.”
Circulation of a Flow Field

CIRCULATION OF A FLOW FIELD

- Flow field may have a “circulation.” The circulation is a very fundamental tool, in theoretical aerodynamics, to calculate aerodynamic lift within an ideal (or potential) flow field.
- The presence of a body (such as an “airfoil”) within a flow field will cause “circulation” within the flow field. If we calculate the amount of circulation within the flow field, we can calculate the amount of “lift” of the body.
- Mathematically, the circulation can be calculated by either a line integral over a given line path \((C)\) defined by the presence of the body, or an area integral over the area \((S)\) bounded by the line path.

\[
\Gamma = -\oint_C \mathbf{V} \cdot d\mathbf{s} = -\iint_S (\nabla \times \mathbf{V}) \cdot d\mathbf{S} \tag{circulation}
\]

CIRCULATION THEORY: CLASSICAL THEORY OF AERODYNAMICS

- The “circulation theory” is a classical mathematical formulation for the calculation and the aerodynamic lift of a body within a given flow field.
- This is very “well-established” mathematical model based flow field analysis of an ideal (or potential) flow field (pure theory basis).
- In circulation theory, an aerodynamic lift is generated due to the time rate of change of airflow in the downward direction, due to the amount of circulation in the flow field, generated by the presence of the body.
- **Kutta-Joukowski Theorem:** (probably) the most important theorem in theoretical aerodynamics
Kutta-Joukowksi Theorem

\[ L' = \rho_\infty V_\infty \Gamma \]

KUTTA-JOUKOWSKI THEOREM
Class Example Problem B-2-1
Related Subjects . . . “Circulation”

Consider the 2-D velocity field:
\[ \mathbf{V} = \frac{y}{(x^2 + y^2)} \mathbf{i} - \frac{x}{(x^2 + y^2)} \mathbf{j} \]

If the velocity is given in units of ft/s, calculate the circulation around a circular path of radius \( r \). centered around the origin.
Homework B-2-1a

Consider a 2-D flow field: \( \vec{V} = 2x \hat{x} + 2y \hat{y} \) defined within the 2-D Cartesian geometry.

Calculate the circulation of the flow field, using the following two methods:

(a) \( \Gamma = -\int_C \vec{V} \cdot d\vec{s} \)

(b) \( \Gamma = -\iint_S (\nabla \times \vec{V}) \cdot d\vec{S} \)
Homework B-2-1b

(a) As we learned in class, a streamline is a curve whose tangent at any point is in the direction of the velocity vector at that point. Mathematically, this means: \( d\mathbf{s} \times \mathbf{V} = 0 \). Starting from this definition of streamline, derive three equations of streamline (in \( x\)-\( y \), \( y\)-\( z \), and \( x\)-\( z \) planes) as:

\[
\frac{dy}{dx} = \frac{v}{u}, \quad \frac{dy}{dz} = \frac{v}{w}, \quad \text{and} \quad \frac{dx}{dz} = \frac{u}{w}
\]

(b) As we learned in class, the angular velocity can be related to the rate of shear strain. Starting from the angular velocity in \( x\)-\( y \) plane:

\[
\omega_z = \frac{1}{2} \left( \frac{\Delta \theta_1}{\Delta t} + \frac{\Delta \theta_2}{\Delta t} \right)
\]

Derive the angular velocity of a fluid element in \( x\)-\( y \) plane:

\[
\omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)
\]
### Stream Function (1)

#### STREAM FUNCTION

- Staring from the equation of streamlines (in $x$-$y$ plane), let’s derive the stream function.
  
  \[
  \frac{dy}{dx} = \frac{v}{u} \implies u(x, y)dy = v(x, y)dx \implies \int u(x, y)dy = \int v(x, y)dx
  \]
  
  \[
  \implies f(x, y) = c
  \]

- Let us denote this function $f(x, y)$ by the symbol $\psi$, called the **stream function**. A “unique” equation for a streamline is given by setting the stream function equal to a “unique” constant.

  $\psi = c_1$, $\psi = c_2$, $\psi = c_3$, $\ldots$ represent the streamlines of the flow field.

#### STREAMLINES OF A FLOW FIELD

If $u$ and $v$ are known functions of $x$ and $y$, then the equation of streamlines can be integrated to yield:

\[
f(x, y) = c \quad \text{where, } c \text{ is an arbitrary constant (will be different value for each streamline)}.
\]

- $f(x, y) = c$ is called, the **equation of streamlines**. Let’s properly understand that this generates “multiple” equations (a uniquely different equation for each streamline, for a different value of $c$).

- $f(x, y) = \psi$ is called, the **stream function** (a function, that can generate multiple “equations of streamlines” with multiple constants $c$ assigned to this function).
Stream Function (2)

1. \( \psi = \text{constant} \) gives the equation of a streamline.
2. The stream function can be defined in 2-D flows only.
3. If the stream function is known, the flow field can be described by:
   \[
   u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad V_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad V_\theta = -\frac{\partial \psi}{\partial r}
   \]

CONCEPT OF STREAM FUNCTION

Stream function \( \bar{\psi} \) is a mathematical function that can represent a 2-D \textbf{steady} flow field.

For \textbf{incompressible} flow field: \( \psi = \bar{\psi}/\rho \) can be defined as the stream function.

- The dimension of stream function is:
  \( \bar{\psi} \) kg/(s·m): mass flow rate per unit depth.
  \( \psi \) m\(^3\)/(s·m): volume flow rate per unit depth.

- The difference in stream function represents:
  \( \Delta \bar{\psi} \) : mass flow (per unit depth) between two adjacent streamlines.
  \( \Delta \psi \) : volume flow (per unit depth) between two adjacent streamlines.
Velocity Potential (1)

\[ \vec{V} = \nabla \phi \]

\[ u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z} \]

\[ V_r = \frac{\partial \phi}{\partial r}, \quad V_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad V_z = \frac{\partial \phi}{\partial z} \]

VELOCITY POTENTIAL (1)

- Recall that an irrotational flow field is defined as a flow where the vorticity is zero at every point of the flow field, such that:
  \[ \vec{\xi} = \text{curl} (\vec{V}) = \nabla \times \vec{V} = 0 \]

- Consider a scalar function \( \phi \), called the velocity potential.
- The vector identity can be written as:
  \[ \nabla \times (\nabla \phi) = 0 \]

- This equation states that “for an irrotational flow, there exists a scalar function \( \phi \) such that the velocity is given by the gradient of \( \phi \).”

EQUIPOTENTIAL LINES OF A FLOW FIELD

Similar (in concept) to streamlines, the equation of equipotential lines can be given as:

\[ f(x, y) = c \]

where, \( c \) is an arbitrary constant (will be different value for each line).

- \[ f(x, y) = c \] is called, the “equation of equipotential lines.” This generates “multiple” equations (a uniquely different equation for each line, for a different value of \( c \)).

- \[ f(x, y) = \phi \] is called, the “velocity potential” (a function, that can generate multiple “equations of equipotential lines” with multiple constants \( c \) assigned to this function).

VELOCITY POTENTIAL (2)

The velocity potential is a function of a spatial coordinate: \( \phi = \phi(x, y, z) \), or \( \phi = \phi(r, \theta, z) \)

For a 3-D Cartesian coordinate system:

\[ \vec{V} = \nabla \phi = u\hat{i} + v\hat{j} + w\hat{k} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \]

Therefore:

\[ u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad \text{and} \quad w = \frac{\partial \phi}{\partial z} \]
**Velocity Potential (2)**

1. \( \phi = \text{constant} \) gives the equation of an equipotential line.
2. The velocity potential can be defined for irrotational flow only.
3. The velocity potential can be defined for both 2-D and 3-D flows.
4. If the velocity potential is known, the flow field can be described by:

\[
\begin{align*}
  u &= \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad w = \frac{\partial \phi}{\partial z} \\
  V_r &= \frac{\partial \phi}{\partial r} \quad V_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \quad V_z = \frac{\partial \phi}{\partial z}
\end{align*}
\]

\[
\frac{dy}{dx}_{\psi=\text{const}} = -\frac{1}{(dy/dx)_{\phi=\text{const}}}
\]

**CONCEPT OF VELOCITY POTENTIAL**

Velocity potential \( \phi \) is a mathematical function that can represent a *steady irrotational* flow field.

If a velocity potential can be defined for a flow field, there will be a significant simplification: Velocity components \((u, v, w)\): 3 unknowns \(\Rightarrow\) velocity potential \(\phi\): 1 unknown.

A flow field that can be described by a velocity potential is called the *potential flows*.

**STREAMLINES AND EQUIPOTENTIAL LINES**

Lines of constant \( \psi \): streamlines
Lines of constant \( \phi \): equipotential lines

The slope of a \( \psi = \text{constant} \) line is the negative reciprocal of the slope of a \( \phi = \text{constant} \) line: means that streamlines and equipotential line are *mutually perpendicular*.

**IDEAL (OR POTENTIAL) FLOW FIELD ANALYSIS**

- The flow field that can define “velocity potential” (steady and irrotational flow field) is called the “potential” flow field.
- In aerodynamics I (AE301), we define “ideal” flow field being “incompressible” “potential” flow field; meaning:
  - (i) steady,
  - (ii) no body force,
  - (iii) inviscid,
  - (iv) irrotational (because inviscid),
  - (v) and incompressible.
- Ideal flow field is governed by the equation, called the “Laplace’s equation.”
- “*Theoretical aerodynamics*” is a basic mathematical analysis, attempting to solve this Laplace’s equation for an ideal flow field.
### Inviscid and Incompressible Flow Equations

**Bernoulli’s Equation:**

\[ p + \frac{1}{2} \rho V^2 = \text{const} \quad \text{along a streamline} \]

**Potential Flow:**

\[ p + \frac{1}{2} \rho V^2 = \text{const} \quad \text{throughout the flow} \]

**Pressure Coefficient:**

\[ C_p = \frac{p - p_\infty}{q_\infty} \quad C_p = 1 - \left( \frac{V}{V_\infty} \right)^2 \]

### EQUATION SIMPLIFICATION PROCESS IN AERODYNAMICS

The \(x\)-momentum equation: **the governing equation**

\[ \rho \frac{Du}{Dt} = - \frac{\partial p}{\partial x} + \rho f_x + (f_x)_{\text{viscous}} \]

\[ dp = -\rho V \, dV \]

\[ \text{Steady, inviscid flow with no body forces: Euler’s equation} \]

\[ p + \frac{1}{2} \rho V^2 = \text{const} \quad \text{along a streamline} \]

\[ \text{Incompressible: Bernoulli’s equation} \]

\[ p + \frac{1}{2} \rho V^2 = \text{const} \quad \text{throughout the flow} \]

\[ \text{Irrotational: ideal (or potential) flows} \]

**VELOCITY-PRESSURE COUPLING FOR AN IDEAL FLOW FIELD**
Laplace’s Equation \((1)\)

\[ \nabla \cdot \mathbf{V} = 0 \quad \nabla^2 \phi = 0 \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \]

**Laplace’s Equation \((1)\)**

Recall, from the conservation of mass (continuity): \(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0\) (differential form)

For steady \(\left( \frac{\partial \rho}{\partial t} = 0 \right)\) and incompressible (\(\rho = \text{constant}\)) flow, this equation can be simplified to:

\(\nabla \cdot \mathbf{V} = 0\) (note: this is the **continuity equation of steady & incompressible flow**)

(Mathematically, this is the “**Divergence of a velocity field**”)

**Laplace’s Equation \((2)\)**

**The Laplacian Operator**

Often, the Laplace’s equation can be expressed by the special operator, called the “**Laplacian**”\((\Delta)\):

\[ \nabla \cdot (\nabla \phi) = \nabla^2 \phi = \Delta \phi = 0 \quad \text{or} \quad \Delta( \phi ) = 0 \] (Laplacian of a scalar function is equal to zero)
POTENTIAL FLOW ANALYSIS

- Potential flow is a flow field that can define a velocity potential.
- Ideal flow field (in aerodynamics I) can be defined as a flow field of steady, inviscid, no body forces, incompressible, and irrotational.
- The ideal flow is governed by the Laplace’s equation. The ideal flow field can be analyzed by pure theory (100% mathematical solution), which is essentially “solving the Laplace’s equation” for a given flow field.

LAPLACE’S EQUATION

- In terms of velocity potential: 3-D Cartesian coordinate system, \( \phi = \phi (x, y, z) \):
  \[
  \nabla^2 \phi = \Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
  \]

- In terms of velocity potential: 3-D cylindrical coordinate system, \( \phi = \phi (r, \theta, z) \):
  \[
  \nabla^2 \phi = \Delta \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
  \]

- In terms of stream function: 2-D Cartesian coordinate system, \( \psi = \psi (x, y) \):
  \[
  \nabla^2 \psi = \Delta \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0
  \]

In 2-D polar coordinate system, \( \psi = \psi (r, \theta) \):
\[
\nabla^2 \psi = \Delta \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0
\]
Laplace’s equation is:

1. Partial differential equation (PDE)
2. With 2nd order
3. And linear

The Laplace’s equation is a **linear 2nd order elliptic PDE**. The distinctive features include:

- Laplace’s equation is the governing equation for an **ideal flow field**. Therefore, one can analyze flow field behavior (i.e., velocity, pressure, aerodynamic forces & moments) of an ideal flow field by “mathematically solve” the Laplace’s equation. This is the very foundation of aerodynamics (**theoretical aerodynamics**).

- In AE301, we’ll start with very simple ideal flow field analysis (**potential flow theory**): flow over a circular cylinder without and with spin (circulation).

- Hopefully, later, you’ll start using the aerodynamic analysis software (**XFOIL / XFLR5**). Note that the software is based entirely on the potential flow theory. **Understanding the basic mechanism of aerodynamic analysis (the potential flow theory) is essentially understanding the theory of aerodynamics.**

- **Using the software AS A BLACK BOX is NOT an engineering practice!** Anyone can do such a thing . . . the purpose of AE301 is to understand “how the software works,” and not “using the software as a black box.”
Homework B-2-2a

As we learned in class, the definition of pressure coefficient \( (C_p) \), at a given point, is: 
\[
C_p = \frac{p - p_e}{q_\infty} 
\]
where,
- \( p \): Static pressure of a given point
- \( p_e \): Freestream static pressure
- \( q_\infty \): Freestream dynamic pressure \( \left( q_\infty = 0.5 \rho_\infty V_\infty^2 \right) \)
- \( \rho_\infty \): Freestream density
- \( V_\infty \): Freestream velocity

Starting from this definition of pressure coefficient, derive the pressure coefficient of potential flow: 
\[
C_p = 1 - \left( \frac{V}{V_\infty} \right)^2
\]
Homework B-2-2b

Consider a velocity field: \( \mathbf{V} = c \hat{r} \) (c is a constant).

(a) Calculate the equation of streamlines, which are passing through \((x, y) = (0, 2)\) and \((x, y) = (5, 0)\).

(b) Sketch these two streamlines.

(c) Calculate the vorticity of this velocity field.

Hints . . .
- First, you must transform velocity field from 2-D polar to Cartesian coordinate system.
- Apply the equation of stream line.
- Two equations of streamlines need to be determined by applying specific boundary condition each: ones that passing through \((x, y) = (0, 2)\) and \((x, y) = (5, 0)\).

- It is easier to calculate vorticity in 2-D polar coordinate system: \( \nabla \times \mathbf{V} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r \hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix} \)
Homework B-2-2c

(a) Derive the Laplace’s equation $\nabla^2 \phi = 0$ from the continuity: $\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0$. List all assumptions and show all details of your derivation. Otherwise, no partial credits.

(b) Alternatively, the Laplace’s equation can also be derived from stream functions: $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ with an assumption of irrotational flow. Derive this alternative form of Laplace’s equation: $\nabla^2 \psi = 0$.

Hints . . .
• (a) Review your notes! Understanding the concepts is the most important fundamental task for each homework assignment!
• (b) In 2-D Cartesian coordinate system, the stream function ($\psi$) can be defined as: $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$.

Also, for irrotational flow: $\nabla \times \mathbf{V} = 0$. Combine these two.
Homework B-2-2d

Answer the followings (explain in your own words).
(a) What is the difference between “rotational” and “irrotational” flow?
(b) Explain the basic concept of “circulation theory” of classical aerodynamics.
(c) What is Kutta-Joukowski theorem?
(d) What is “stream function”?
(e) What is “velocity potential”? 