Unit C-4: List of Subjects

- Thin Airfoil Approximation
- Thin Airfoil Theory
- Symmetric Thin Airfoil
- Cambered Thin Airfoil
- Real Flow over an Airfoil
THIN AIRFOIL APPROXIMATION

If the airfoil is thin, a single vortex sheet can be used to approximate a thin airfoil by replacing it along the camber line: **thin airfoil theory**.

PRANDTL’S CLASSICAL THIN AIRFOIL THEORY

The classical Prandtl’s thin airfoil theory is covered here:

- Rather than modeling the mean camber line by vortex sheet, the **chord line** (the straight line connecting between leading edge and trailing edge) is represented by a vortex sheet as a function of straight horizontal coordinate system \((x)\): the vortex sheet is placed on the chord line: \(\gamma = \gamma(x)\)

- The camber line is forced to be a streamline, and \(\gamma\) is calculated to satisfy this condition as well as Kutta condition: \(\gamma(c) = 0\)

- For the camber line to be a streamline, the normal component of freestream is going to be cancelled by the downwash generated by the vortex sheet (everywhere along the mean camber line); mathematically: \(V_{\infty,n} + w'(s) = 0\)
At any point \( P \) on the camber line, where the slope of the camber line is \( dz/dx \),
\[
V_{\infty,n} = V_{\infty} \sin[\alpha + \tan^{-1}(-dz/dx)]
\]

For a thin airfoil, both \( \alpha \) and \( \tan^{-1}(-dz/dx) \) are small values, and for a small angle: \( \sin \theta \approx \tan \theta \approx \theta \), so \( V_{\infty,n} = V_{\infty}[\alpha - dz/dx] \)

The velocity \( dw \) at location \( x \) induced by the small elemental vortex \( d\xi \) at location \( \xi \) is given by:
\[
dw = -\frac{\gamma(\xi)d\xi}{2\pi(x-\xi)}
\]

Thus, the velocity \( w(x) \) induced at location \( x \) by all the elemental vortices along the chord line can be obtained by integrating \( dw \) from the leading edge \( (\xi = 0) \) to the trailing edge \( (\xi = c) \) as:
\[
w(x) = -\int_{0}^{c} \frac{\gamma(\xi)d\xi}{2\pi(x-\xi)}
\]
Recall, for the camber line to be streamline:

\[V_{x,n} + w'(s) = 0\]

where \( w'(s) \approx w(x) \) for thin airfoil, so

\[V_\infty \left( \alpha - \frac{dz}{dx} \right) - \int_0^c \frac{\gamma(\xi) d\xi}{x - \xi} = 0\]

\[
\frac{1}{2\pi} \int_0^c \frac{\gamma(\xi) d\xi}{x - \xi} = V_\infty \left( \alpha - \frac{dz}{dx} \right)
\]

This is the governing equation of thin airfoil theory.
**Symmetric Thin Airfoil (1)**

![Graph](image)

**SYMmetric Thin Airfoil Solutions (1)**

For symmetric airfoil: \(dz/dx = 0\), therefore:

\[
\frac{1}{2\pi} \int_{0}^{c} \frac{\gamma(\xi)d\xi}{x-\xi} = V_{\infty}\alpha
\]

Transformation from \(\xi\) to \(\theta\):

\[\xi = \frac{c}{2}(1-\cos\theta), \quad d\xi = \frac{c}{2}\sin\theta d\theta\]

Transformation from \(x\) to \(\theta_{0}\):

\[x = \frac{c}{2}(1-\cos\theta_{0})\]

\[
\frac{1}{2\pi} \int_{0}^{\theta_{0}} \gamma(\theta)\sin\theta d\theta
\]

The mathematical solution to this equation is:

\[
\gamma(\theta) = 2\alpha V_{\infty} \frac{(1+\cos\theta)}{\sin\theta}
\]
SYMMETRIC THIN AIRFOIL SOLUTIONS (2)

The total circulation around the airfoil is: \[ \Gamma = \int \gamma(\xi)d\xi = \frac{c}{2} \int \gamma(\theta)\sin\theta d\theta \]

Solution of the thin airfoil theory is: \[ \Gamma = \alpha cV_\infty \int_0^\pi (1 + \cos \theta)d\theta = \pi \alpha cV_\infty \]

LIFT AND MOMENT COEFFICIENTS

From the Kutta-Joukowski theorem: \[ L' = \rho_\infty V_\infty \Gamma = \pi \alpha \rho_\infty V_\infty^2 \]

The lift coefficient: \[ c_l = \frac{L'}{q_\infty c} = \frac{\pi \alpha \rho_\infty V_\infty^2}{\frac{1}{2} \rho_\infty V_\infty^2 c} = \frac{2\pi\alpha}{c} \]

Moment at leading edge: \[ M'_{\text{LE}} = -\int_0^c \xi dL = -\rho_\infty V_\infty \int_0^c \xi \gamma(\xi) d\xi \]

=> through transformation and integration: \[ M'_{\text{LE}} = -q_\infty c^2 \frac{\pi\alpha}{2} \]

The moment coefficient: \[ c_{m,\text{le}} = \frac{M'_{\text{LE}}}{q_\infty c^2} = -\frac{\pi\alpha}{2} = \frac{-c_l}{4} \]

The moment coefficient at quarter chord point (recall, from Unit A-5):
\[ c_{m,c/4} = c_{m,\text{le}} + \frac{c_l}{4} = \frac{-c_l}{4} + \frac{c_l}{4} = 0 \]

CENTER OF PRESSURE AND AERODYNAMICS CENTER LOCATIONS

The location of center of pressure is: \[ \bar{x}_{cp} = \frac{1}{4} - \frac{c_{m,c/4}}{c_l} = \frac{1}{4} - \frac{0}{2\pi\alpha} = \frac{1}{4} \]

The location of aerodynamic center is: \[ \bar{x}_{ac} = -\frac{m_0}{a_0} + \frac{1}{4} = -\frac{0}{2\pi} + \frac{1}{4} = \frac{1}{4} \]

- For a symmetric thin airfoil, the quarter-chord point is both the center of pressure and the aerodynamic center.
Consider a symmetric thin airfoil (such as NACA 0006) at 4 degrees angle of attack. Using the thin airfoil theory, calculate the followings and compare them against NACA airfoil data:
(a) The lift coefficient
(b) The moment coefficient at the quarter chord

Using the thin airfoil theory:

At 4 degrees AOA: 4 degrees = \(4(\pi/180)\) = 0.06981 rad

(a) \(c_l = 2\pi\alpha = 2\pi(0.06981) = 0.4386\)

(b) \(c_{m,c/4} = 0\)
From NACA 0006 airfoil data, at 4 degrees AOA: 4 degrees:

(a) $c_l = 0.4$

(b) $c_{m,c/4} = 0$
Homework C-4-1a

Answer the followings. **Explain in your own words.**

Source or vortex panel method, thin airfoil theory (2-D), and lifting line theory (3-D) are all based on potential flow analysis (thus, all “theoretical” aerodynamics).

(a) (Once again) List (at least) four assumptions (thus, the limitation of applications) for these theoretical aerodynamics tools (this is quite important).

(b) What are the differences between vortex panel method and thin airfoil theory.
**Homework C-4-1b**

The mathematical solutions of a symmetrical thin airfoil theory (through coordinate transformation) are given below:

\[
\Gamma = \frac{\pi \alpha V_c}{\gamma} \int_0^\infty (1 + \cos \theta) d\theta = \pi \alpha c V_c
\]

\[
M_{LE} = -\int_0^\infty \xi(\xi)(d\xi) = -\rho \pi V_c \int_0^\infty \xi \gamma(\xi) d\xi = -q_c c^2 \frac{\pi \alpha}{2}
\]

Determine the followings:

(a) Section Lift Coefficient (in terms of angle of attack: \(\alpha\)): \(c_l = ?\)

(b) Section Moment Coefficient at the Leading Edge (in terms of lift coefficient: \(c_l\)): \(c_{m,le} = ?\)

(c) Section Moment Coefficient at the Quarter Chord Location: \(c_{m,c/4} = c_{m,le} + (c_l/4) = ?\)

(d) Location of “Center of Pressure”: \(\bar{x}_{cp} = \frac{1}{4} - \frac{c_{m,c/4}}{c_l} = ?\)

(e) Location of “Aerodynamic Center”: \(\bar{x}_{ac} = \frac{-m_0}{\alpha_0} + \frac{1}{4} = ?\)
CAMBERED THIN AIRFOIL SOLUTIONS

The generalization of the methods for a thin symmetric airfoil $(dz/dx \neq 0)$ yields:

$$
\frac{1}{2\pi} \int_{0}^{\pi} \gamma(\xi)d\xi = V_{\infty} \left( \alpha - \frac{dz}{dx} \right)
$$

The mathematical solution to this equation is:

$$
\gamma(\theta) = 2V_{\infty} \left( A_{0} \frac{1 + \cos \theta}{\sin \theta} + \sum_{n=1}^{\infty} A_{n} \sin n\theta \right)
$$

The coefficients:

$$
A_{0} = \alpha - \frac{1}{\pi} \int_{0}^{\pi} \frac{dz}{d\theta_{0}} \quad A_{n} = \frac{2}{\pi} \int_{0}^{\pi} \frac{dz}{d\theta_{0}} \cos n\theta_{0} d\theta_{0}
$$
LEADING EDGE STALL

Example: NACA 4412

Characteristics of relatively thin airfoils with thickness ratios between 10 and 16 percent of the chord length.

- Post-stall characteristics: **rapid loss of the lift**
EFFECTS OF AIRFOIL THICKNESS ON STALL

Thin airfoil: thin airfoil is desired for high-speed applications (from high transonic to supersonic), in order to minimize profile and wave drags.

Thick airfoil: thick airfoil is desired for low-speed applications (low subsonic), due to the favorable stall characteristics.