Newtonian Gravity

"Newtonian gravity" is the name we give to the model we will use to express the behavior of objects in a gravitational field. The Newtonian model does not include any "Einsteinian" effects. Namely, we assume objects are moving relatively slowly \( v \ll c \) and that the gravitational field is fairly weak. For example, Newtonian Gravity is sufficient for all calculations involving gravity in our solar system, unless extreme accuracy is needed (GPS satellites, atomic clock synchronization, etc.)

Newton's theory of gravity

In Cartesian co-ords:
\[
\vec{r} = (x, y, z) \\
\vec{r}' = (x', y', z')
\]

Newton's 2nd Law
\[
\vec{F} = \frac{d\vec{p}}{dt}
\]

where
\[
\vec{F} = -G \frac{mM}{r^2} \hat{r}
\]

\( \hat{r} \) = \( \vec{r} - \vec{r}' \)

\( r = |\vec{r}| \) , \( \hat{r} = \frac{\vec{r}}{r} \)

\( \vec{p} = m \vec{\dot{r}} \)

For any quantity that depends on time, "dot" means "time-derivative of," e.g. \( \vec{a} = \frac{d\vec{v}}{dt} \).
Example: Acceleration of one point mass towards another

\[ \vec{r} = \frac{GMm}{r^2} \]

Find the acceleration of \( m \):

\[ \hat{\vec{F}} = -G \frac{mmM}{r^2} \hat{r} \]

\[ \dot{\vec{p}} = \frac{d}{dt} (m \vec{v}) = m \dot{\vec{v}} = m \ddot{\vec{a}} \]

\[ \Rightarrow \vec{a} = -G \frac{M}{r^2} \hat{r} \]

Direction of the acceleration \( \hat{\vec{a}} \) is \( \hat{r} \), i.e. towards \( M \).

Magnitude \( |\vec{a}| \) of the acceleration \( \vec{a} \) is \( |\vec{a}| = \frac{GM}{r^2} \)

Note: In the cases where the motion is along a single line (as in this case) it's useful to define the "signed magnitude" of the position, velocity, acceleration, etc.

e.g. The signed magnitude of \( \hat{\vec{a}} \) is \( \alpha \)

\[ \alpha = \begin{cases} \frac{1}{2} |\vec{a}| & \text{when } \hat{\vec{a}} \text{ points in the direction defined as "positive" (e.g. } \hat{x} \text{)} \\ -|\vec{a}| & \text{when } \hat{\vec{a}} \text{ points in the opposite direction. (e.g. } -\hat{x} \text{)} \end{cases} \]

For example, if we choose the origin at \( M \), and let the particles lie on the \( x \)-axis:

\( \vec{a} = -\frac{GMM}{x^2} \text{sign}(x) \)
Gravitational Field

According to Newton's law of gravitation, the force on a test mass \( m \) due to a point source mass \( M \) is

\[
\vec{F} = m \left( \frac{G M}{r^2} \right) \hat{r} = \vec{g}
\]

Define everything except \( m \) to be the gravitational field \( \vec{g} \) in which \( m \) lies.

The fact that forces from different source masses must add like vectors implies that the fields add like vectors.

\[
\vec{F} = \vec{F}_1 + \vec{F}_2
\]

\[
\Rightarrow \quad m \vec{g} = m \vec{g}_1 + m \vec{g}_2
\]

Net (total) force

\[
\vec{F} = \vec{F}_1 + \vec{F}_2
\]

Generalizing to an arbitrary number \( N \) source masses

\[
\vec{g} = \sum_{i=1}^{N} \vec{g}_i = G \sum_{i=1}^{N} \frac{M_i}{r_i^2} \hat{r}_i
\]
What is the force on $m$?

\[ \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = -G \left( \frac{M_1}{r_1^2} \hat{r}_1 + \frac{M_2}{r_2^2} \hat{r}_2 \right) \]

\[ \hat{r}_1 = \frac{\mathbf{r}_1 - \mathbf{r}_1'}{r_1} = \frac{(2 \text{ meter}) \hat{x} - (1 \text{ meter}) \hat{y}}{(3 \text{ meter}) \hat{y}} = \frac{(3 \text{ meter}) \hat{y} - (3 \text{ meter}) \hat{y}}{3 \text{ meter}} = \hat{y} \]

\[ \hat{r}_2 = \frac{\mathbf{r}_2 - \mathbf{r}_2'}{r_2} = \frac{(2 \text{ meter}) \hat{x} - (1 \text{ meter}) \hat{y}}{(1 \text{ meter}) \hat{y}} = (1 \text{ meter}) \hat{y} \]

\[ \mathbf{F} = G \left( \frac{M_1}{r_1^2} \hat{r}_1 + \frac{M_2}{r_2^2} \hat{r}_2 \right) = -G \left( \frac{M_1}{r_1^2} \hat{y} + \frac{M_2}{r_2^2} \hat{y} \right) \]

Now plug in the numbers...
\[ \ddot{g} = -6.67 \times 10^{-11} \text{meters}^3 \text{kg}^{-1} \text{s}^{-2} \left( \frac{1000 \text{kg}}{(3 \text{ meters})^2} + \frac{1000 \text{ kg}}{(1 \text{ meter})^2} \right) \hat{y} \]

\[ = \left( 7.4 \times 10^{-8} \frac{\text{meters}}{\text{s}^2} \right) (-\hat{z}) \text{ Newton/kg} \]

\[ = (-\hat{z}) 0.074 \mu \text{N/kg} \]

\[ \vec{F} = m \ddot{g} = (100 \text{ kg}) (0.074 \mu \text{N/kg}) = 7.4 \mu \text{N} \] (A pin weighs about 100x this much)

Standard way of writing this out

\[ \ddot{g} = \ddot{g}_1 + \ddot{g}_2 = -\ddot{g} \frac{M_1}{r_1^2} \hat{y} - \ddot{g} \frac{M_2}{r_2^2} \hat{y} \]

\[ = \left( -\ddot{g} \frac{M_1}{r_1^2} - \ddot{g} \frac{M_2}{r_2^2} \right) \hat{y} \]

Now put in #’s.

However you choose to do this, don’t forget to draw the diagram!
Example

\[ \hat{r}'_1 = -\hat{y} \]
\[ \hat{r}'_2 = \hat{y} \]
\[ \hat{r}_1 = \hat{r}'_1 - \hat{r}'_2 = \hat{z} - (-\hat{y}) = \hat{y} + \hat{z} \]
\[ \hat{r}_2 = \hat{r}'_1 - \hat{r}'_2 = \hat{z} - \hat{y} = -\hat{y} + \hat{z} \]

\[ \Rightarrow \hat{r}_1^2 = \hat{r}_1 \cdot \hat{r}_1 = (\hat{y} + \hat{z}) \cdot (\hat{y} + \hat{z}) = \hat{y} \cdot \hat{y} + 2 \cdot \hat{y} \cdot \hat{z} + \hat{z} \cdot \hat{z} \]

(both zero since \( \hat{y} \perp \hat{z} \))

\[ \hat{r}_2^2 = 2 \]

\[ \Rightarrow \begin{cases} \hat{r}'_1 = \sqrt{2} (\hat{y} + \hat{z}) \\ \hat{r}'_2 = \frac{1}{\sqrt{2}} (-\hat{y} + \hat{z}) \end{cases} \]

\[ \hat{r}_1 = \hat{r}'_1 + \hat{r}'_2 = -\mathcal{G} \frac{M_1}{\hat{r}_2^2} \hat{y} - \mathcal{G} \frac{M_2}{\hat{r}_2^2} \hat{z} \]

(substitute in for \( \hat{r}_1 \) & \( \hat{r}_2 \))

\[ = -\mathcal{G} \frac{M_1}{\hat{r}'_2^2 \sqrt{2}} \hat{y} - \mathcal{G} \frac{M_1}{\hat{r}'_1^2 \sqrt{2}} \hat{z} + \mathcal{G} \frac{M_2}{\hat{r}'_2^2 \sqrt{2}} \frac{1}{2} \hat{z} - \mathcal{G} \frac{M_2}{\hat{r}'_2^2 \sqrt{2}} \frac{1}{2} \hat{y} \]

\[ = \left( \frac{-1}{\sqrt{2}} \mathcal{G} \frac{M_1}{\hat{r}'_2^2} + \frac{1}{2} \mathcal{G} \frac{M_2}{\hat{r}'_2^2} \right) \hat{y} - \left( \frac{1}{\sqrt{2}} \mathcal{G} \frac{M_1}{\hat{r}'_1^2} + \frac{1}{2} \mathcal{G} \frac{M_2}{\hat{r}'_2^2} \right) \hat{z} \]

\[ = \frac{\mathcal{G}}{\sqrt{2}} \left[ \left( \frac{M_2}{\hat{r}'_2^2} - \frac{M_1}{\hat{r}'_1^2} \right) \hat{y} - \left( \frac{M_1}{\hat{r}'_1^2} + \frac{M_2}{\hat{r}'_2^2} \right) \hat{z} \right] \]

Now put in #'s
Since \( M_1 = M_2 = M \) \& \( r_1 = r_2 \equiv r = \sqrt{2} \) meters

\[
\vec{g} = \frac{G}{\sqrt{2}} \left[ \left( \frac{M}{r^2} - \frac{M}{r^2} \right) \hat{y} - \left( \frac{M}{r^2} + \frac{M}{r^2} \right) \hat{z} \right] \frac{1}{\text{meter}^2}.
\]

\[
= - \frac{G}{\sqrt{2}} \frac{(2M)^2}{2 \text{ meter}^2} = - \frac{GM}{\sqrt{2}} \frac{\hat{z}}{\text{meter}^2}
\]

\[
= 6.67 \times 10^{-11} \frac{\text{meter}^3}{\text{kg} \cdot \text{s}^2} \times 1000 \text{ kg} \frac{\hat{z}}{\sqrt{2} \text{ meter}^2}
\]

\[
= 4.7 \times 10^{-8} \frac{\text{meter}}{s^2} \hat{z}
\]

\[
\Rightarrow \quad \vec{F} = m \vec{g} = (100 \text{ kg}) \left( 4.7 \times 10^{-8} \frac{\text{meter}}{s^2} \right) \hat{z}
\]

\[
= 4.7 \text{ mN} \hat{z}
\]

**Easier way:** Note that by symmetry \( y \)-components cancel, so only need to find \( z \)-comp.'s which are equal

\[
(\vec{g}_1)_z = - \frac{GM}{r^2} \cos(45^\circ) = - \frac{GM}{\sqrt{2} r^2}
\]

Since \( (\vec{g}_1)_z = (\vec{g}_2)_z \)

Total \( z \)-comp. is \( \vec{g}_z = 2(\vec{g}_1)_z = - \frac{\sqrt{2} GM}{r^2} \)

\[
\Rightarrow \quad \vec{g} = \frac{\sqrt{2} GM}{r^2} \hat{z} = 4.7 \frac{\text{meter}}{s^2} \hat{z} \quad \text{as before}
\]
Continuous Source Mass

\[ \mathbf{\tilde{g}} = \int \mathbf{d}\vec{g} \]
Source mass

\[ \mathbf{\tilde{g}} = -G \int \frac{dM}{r^2} \mathbf{r} \]
Source mass

We will use one of these two forms in this class.

\[ \Rightarrow \mathbf{\tilde{g}} = -G \int \int \int \frac{\mathbf{r} \cdot \mathbf{r}'}{r'^3} \, dV \]
Source mass

This form would be the one to use for numerical computation of the \( \mathbf{\tilde{g}} \)-field due to an arbitrary object.

Volume integral over the
priced coordinates,

Recall:
\[ \mathbf{r}' = \mathbf{r} - \mathbf{r} = (x-x') \hat{x} + (y-y') \hat{y} + (z-z') \hat{z} \]
\[ \Rightarrow r'^3 = (r^3 - r)^3 = (r^3 - r)^3 = (r^3 + r^3 - 2rr)^3 \]

Example

Gravitational field from a line of mass

\[ \mathbf{\tilde{g}} = \int \mathbf{d}\vec{g} \]
field point

All horizontal components must cancel with an equal but opposite component due to the wire on the far side of the field point.

\[ x \text{-sec area} = A \]

\[ dM = \pi A dx \]

\[ d\mathbf{g}_z = d\mathbf{g} | \cos \theta = \frac{(GdM)}{r^2} \frac{z}{r} = \pi A z \frac{dx}{(x^2 + z^2)^{3/2}} \]

Only need to consider \( z \) component since \( d\mathbf{g} = (-\mathbf{\hat{z}}) \, d\mathbf{g}_z \)
\[ q = \int_0^\infty d\vec{q} = -\hat{z} \int_0^\infty d\hat{q}_z = -\hat{z} \int_0^\infty \frac{d\hat{x}}{(\hat{x}^2 + \hat{z}^2)^{3/2}} \]

Change variables \( u = \frac{\hat{x}}{\hat{z}} \Rightarrow \frac{d\hat{x}}{du} = \hat{z} \Rightarrow d\hat{x} = \hat{z} du \)

\[ q = -\hat{z} \int_0^\infty \frac{\hat{z} du}{\hat{z}^2(u^2 + 1)^{3/2}} = -\hat{z} \int_0^\infty \frac{du}{(u^2+1)^{3/2}} \]

Change variables \( \phi = \arctan u \Rightarrow \frac{du}{d\phi} = \frac{1 + \tan^2 \phi}{u^2} \)

\[ q = -\hat{z} \int_{\phi = 0}^{\phi = \pi/2} \frac{d\phi}{\sqrt{1 + \tan^2 \phi}} \]

\[ q = -\hat{z} \int_{\phi = 0}^{\phi = \pi/2} \frac{d\phi}{\cos \phi} \]

(Relevant to challenge problem 13.90 on homework.)
Gravitational Potential

The gravitational potential \( V(\vec{r}) \) is equal to the gravitational potential energy of a test mass with unit mass.

An equivalent way of saying the same thing is that gravitational potential \( V(\vec{r}) \) is the gravitational potential energy of a test mass \( m \), per unit mass of that test mass, namely

\[
V(\vec{r}) = \frac{U(\vec{r})}{m}
\]

where \( U(\vec{r}) \) is the potential energy of the test mass \( m \) (relative to some point \( P_0 \) at which \( U(P_0) = 0 \)).

Main utility of \( V(\vec{r}) \) is to find the gravitational field \( \vec{g}(\vec{r}) \) in an easier way than by the methods used so far. If we know \( V(\vec{r}) \), then it can be shown (using vector calculus) that "partial derivatives"

\[
\vec{g}(\vec{r}) = -\vec{\nabla} V(\vec{r}) = - \begin{bmatrix} \frac{\partial V(\vec{r})}{\partial x} + \hat{y} \frac{\partial V(\vec{r})}{\partial y} + \hat{z} \frac{\partial V(\vec{r})}{\partial z} \end{bmatrix}
\]

The operation performed with \( \vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \)

is called the gradient operation. \( \vec{\nabla} \) is called the "gradient operator." The reason for this name is that \( \vec{\nabla} V(\vec{r}) \) points in the direction of steepest ascent \((-\vec{\nabla} V(\vec{r}) \) thus points in the direction of steepest descent.)
on the "surface" \( V(\vec{r}) \), and the magnitude \(| \vec{\nabla} V(\vec{r}) |\) is the slope along that same direction.

So, the gravitational field points in the direction of steepest descent of the surface \( V(\vec{r}) \).

A very handy picture!

So, if we know how to calculate the potential energy \( U(\vec{r}) \) of a test mass, then we can find \( V(\vec{r}) \) and by taking its gradient, find \( \vec{g}(\vec{r}) \).

But we already know how to calculate the potential energy of something. The potential energy \( U(\vec{r}_i) \) of a test mass (relative to its potential energy at \( \vec{r}_0 \to \infty \)) is just the work required to move the test mass from \( \vec{r}_0 \to \infty \) to \( \vec{r}_i \). \( (U(\vec{r}_0 \to \infty) \) is assumed to be zero.) Namely:

\[
U(\vec{r}_i) = \text{work to bring the mass to } \vec{r}_i \text{ from infinity}
\]

"Path integral" \( \Rightarrow \int \vec{E}(\vec{r}, \vec{r}_i) \cdot d\vec{l} \) we must consider that any force, i.e., apply an equal and opposite force to get in moving against the field.
\[ U(r) = - \int_{L(r_0, r)} \vec{F} \cdot d\ell \]

Result is path-independent

Any path from \( r_0 \) (infinity) will give same answer!

Case of a point source mass

For a point source mass \( F = -G \frac{Mm}{r^2} \hat{r} \)

Choose a radial path \( L(r_0, r) \)

\[ \vec{F} \cdot d\ell = -G \frac{Mm}{r^2} \hat{r} \cdot (-\hat{r}) \frac{dr}{d\ell} \]

\[ = G \frac{Mm}{r^2} dr \]

Limits are from low to high \( r \) as order of path traversal is already accounted for.

\[ \Rightarrow U(r) = - \int_{r_1}^{r} \frac{GMm}{r^2} dr = - \left[ -\frac{GMm}{r} \right]_{r_1}^{r} = -G \frac{Mm}{r} \]
Thus for a point source mass, the potential energy of a test mass $m$ is

$$ U(r) = -\frac{GMm}{r} $$

If the source mass is at the origin $\vec{r}' = 0 \Rightarrow r = |\vec{r}'| = r$

then

$$ U(r) = \frac{GMm}{r}.$$ 

**Case of multiple point source masses**

For multiple point source masses, the potential energy at a test mass is the sum of the potential energies induced by each source mass, namely:

$$ U(r) = \sum_{i=1}^{N} \left( -\frac{GM_{i}m}{|\vec{r} - \vec{r}_i'|} \right) = -mG \sum_{i=1}^{N} \frac{M_{i}}{|\vec{r} - \vec{r}_i'|} $$

**Notes**

1. $U(r)$ is a scalar
2. \( U(\mathbf{r}) \) has a fundamental dependence of \( \frac{1}{r^2} \), as opposed to \( \frac{1}{r^2} \) for \( \mathbf{g} \).

For a cont's source mass

\[
U(\mathbf{r}) = -m \int_{\text{source mass}} \frac{\mathbf{g}(\mathbf{r}')}{r} \, d^3r' 
\]

This integral is usually much easier to do than the similar integral that gave us the field \( \mathbf{g}(\mathbf{r}) \).

Recall that potential \( V(\mathbf{r}) = U(\mathbf{r})/m \).

Thus:

\[
V(\mathbf{r}) \begin{cases} 
- G \frac{M}{r} & \text{for a pt. mass at } \mathbf{r}' \\
- G \sum \frac{M_i}{r_i} & \text{for a collection of pt. masses at } \mathbf{r}_i \\
- G \int \frac{\mathbf{d}M}{r} & \text{for a cont's source mass}
\end{cases}
\]

\( \Rightarrow \) "Easy" to get \( \mathbf{g}(\mathbf{r}) \) using \( \mathbf{g}(\mathbf{r}) = -\nabla V(\mathbf{r}) \), because the integrals above are much easier than those involved in obtaining \( \mathbf{g}(\mathbf{r}) \) directly.
Example

Gravitational field on the axis of a ring of mass with radius $R$. Ring has uniform density $\rho$ and x-sec" area $A$.

\[ \mathrm{d}M = \rho A S \, \mathrm{d}\theta \]

Potential is

\[ V(z) = - \int_0^\theta \frac{\mathrm{d}M}{\rho} = - \rho A S \int_0^\theta \frac{\mathrm{d}\theta}{\sqrt{R^2 + z^2}} \]

But $R$ is constant and $z$ is indep. of $\theta$, so

\[ V(z) = - \frac{\rho A S}{R^2 + z^2} \int_0^\theta \frac{\mathrm{d}\theta}{\sqrt{\frac{R^2}{\theta^2} + \frac{z^2}{\theta^2}}} = -2\pi S A P \frac{G \rho A S}{\sqrt{S^2 + z^2}} \]

\[ \Rightarrow V(z) = - \frac{GM}{\sqrt{S^2 + z^2}} \]

\( \hat{g}(r) = - \nabla V(z) = -\hat{x} \frac{\partial V}{\partial x} - \hat{y} \frac{\partial V}{\partial y} - \hat{z} \frac{\partial V}{\partial z} \)

\[ = \hat{z} \frac{\partial}{\partial z} \left( \frac{GM}{\sqrt{S^2 + z^2}} \right) = (-\hat{z}) \frac{GM^2}{(S^2 + z^2)^{3/2}} \]

Note: As $z \to 0$, $\hat{g} \to 0$. For $\frac{z}{S} \ll 1$, $\hat{g}(z) \approx -\frac{GM}{S^2} \hat{z}$. 
Consider the contribution $dV$ to the potential $V$ at $z$ from the hoop-like element of the disk at radius $s$ with width $ds$. X-sectional area of this hoop is then $A = \pi s ds$. Already calculated the potential contribution from a hoop in the previous example.

So:

$$dV = -\frac{2\pi \rho \pi s A}{\sqrt{s^2 + z^2}} = -2\pi \rho \pi t \frac{s}{\sqrt{s^2 + z^2}} ds$$

Integrate up all contributions $dV$ to get $V$:

$$V(z) = \int_{s=0}^{R} dV = \int_{disk} (-2\pi \rho \pi t \frac{s}{\sqrt{s^2 + z^2}}) ds$$

Then use $\vec{E} = -\nabla V$ to get the field.
Kepler's Laws

\[ \vec{\dot{p}} = m \vec{\dot{r}} \quad \text{and} \quad F = -\frac{GMm}{r^2} \hat{r} \]

to get

\[ m \vec{\dot{r}} = -\frac{GMm}{r^2} \hat{r} \]

and solving for \( \vec{r}(t) \) gives the position of \( m \) as a function of time. (We also need to know the value of \( \vec{r} \) and/or \( \vec{v} \) at specific times to fully solve for \( \vec{r}(t) \) at all times \( t \).)

Considering the general properties of \( \vec{r}(t) \) leads to Kepler's Laws, which are useful relationships that hold for all systems of two gravitationally bound point-masses \( m \) & \( M \) where \( M \gg m \).

Kepler's Laws

1. The orbit of \( m \) is an ellipse one of whose foci is at \( M \).

2. A line joining \( m \) & \( M \) sweeps out equal areas in equal time intervals.

3. \( \tau^2 = a^3 \) where \( \tau \) is the period of \( m \)'s orbit and \( a \) is the semi-major axis of \( m \)'s orbital ellipse.
The field inside and outside a spherically symmetric source mass $s(r^2) = s(r)$.

\[ \mathbf{g}(\mathbf{r}) = \begin{cases} - \frac{G (\frac{4}{3} \pi r^3) m}{r^2} \mathbf{\hat{r}} & \text{for } r < R \\ - \frac{G (\frac{4}{3} \pi R^3) m}{r^2} \mathbf{\hat{r}} & \text{for } r > R \end{cases} \]